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Stability estimate for an inverse wave equation and a multidimensional Borg–Levinson theorem

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ABSTRACT

We consider the stability in an inverse problem of determining the potential q entering the wave equation $\partial_t^2 u - \Delta u + q(x)u = 0$ in a bounded smooth domain of $\mathbb{R}^d$ from boundary observations. The observation is given by a hyperbolic (dynamic) Dirichlet to Neumann map associated to a wave equation. We prove a log-type stability estimate in determining q from a partial Dirichlet to Neumann map provided that q is a priori known in a neighbourhood of the boundary of the spatial domain and satisfies an additional condition. Next, we use this result to establish a stability estimate related to the multidimensional Borg–Levinson theorem.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary Γ. Let $T > 0$ be fixed and let us denote $(0,T) \times \Omega$ and $(0,T) \times \Gamma$ respectively by Q and Σ. We set $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d;\ |x| = 1\}$ and we consider the following initial–boundary value problem (IBVP in short) for the wave equation:

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = 0 & \text{in } Q, \\ u(0,\cdot) = 0, \quad \partial_t u(0,\cdot) = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma. \end{cases} \tag{1.1}$$

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Let

$$H^{1,1}(\Sigma)=L^2\big(0,T;H^1(\Gamma)\big)\cap H^1\big(0,T;L^2(\Gamma)\big).$$

We prove (see Theorem A.3 in Appendix A) that for any $q\in L^\infty(\Omega)$ and $f\in H^{1,1}(\Sigma)$ the IBVP (1.1) has a unique solution

$$u_q\in\mathcal{C}\big([0,T];H^1(\Omega)\big)\cap\mathcal{C}^1\big([0,T];L^2(\Omega)\big)$$

such that $\partial_\nu u_q\in L^2(\Sigma)$. In addition, for any positive constant M, there exists a positive constant C, depending only on Ω, T and M, such that for all $q\in L^\infty(\Omega)$ with $\|q\|_{L^\infty(\Omega)}\leqslant M$, the following estimate holds

$$\|u\|_{\mathcal{C}([0,T];H^1(\Omega))}+\|u_q\|_{\mathcal{C}^1([0,T];L^2(\Omega))}+\|\partial_\nu u_q\|_{L^2(\Sigma)}\leqslant C\|f\|_{H^{1,1}(\Sigma)}.$$

In particular the following operator, usually called the Dirichlet to Neumann map (DN map in short),

$$\begin{aligned}\Lambda_q:H^{1,1}(\Sigma)&\to L^2(\Sigma)\\ f&\mapsto\Lambda_q(f)=\partial_\nu u_q\end{aligned}\tag{1.2}$$

is bounded.

Let Γ_0 be an arbitrary non-empty relatively open subset of Γ. We set $\Sigma_0=(0,T)\times\Gamma_0$ and we introduce the partial DN map defined as follows

$$\begin{aligned}\Lambda_q^\sharp:H^{1,1}(\Sigma)&\to L^2(\Sigma_0)\\ f&\mapsto\Lambda_q^\sharp(f)=\partial_\nu u_q|_{\Sigma_0}.\end{aligned}\tag{1.3}$$

We observe that since Λ_q is bounded, $\Lambda_q^\sharp$ is also bounded.

We establish a stability result for the inverse problem consisting in the determination of the potential q from the partial DN map $\Lambda_q^\sharp$. Let $\mathcal{C}^{0,\mu}(\overline{\Omega})$ be the usual Hölder space with $0<\mu<1$. We fix $q_0\in\mathcal{C}^{0,\mu}(\overline{\Omega})$ and we consider the set

$$\mathfrak{X}(M,\omega)=\big\{q\in\mathcal{C}^{0,\mu}(\overline{\Omega});\ \|p\|_{L^\infty(\Omega)}\leqslant M,\ q(x)=q_0(x)\text{ in }\omega\big\},\tag{1.4}$$

where $\omega\subset\Omega$ is an arbitrary neighbourhood of Γ and M is a given positive constant.

The first main result in this paper can be stated as follows.

Theorem 1.1. *There exist $C>0$, $\delta\in(0,1)$ and sufficiently large T such that*

$$\|q_1-q_2\|_{H^{-1/2}(\Omega)}\leqslant C\big(\big\|\Lambda_{q_1}^\sharp-\Lambda_{q_2}^\sharp\big\|^\delta+\big|\log\big(\big\|\Lambda_{q_1}^\sharp-\Lambda_{q_2}^\sharp\big\|\big)\big|^{-\delta}\big)\tag{1.5}$$

for any $q_1,q_2\in\mathfrak{X}(M,\omega)$.

If in addition $q_1,q_2\in H^s(\Omega)$ for $s>d/2$ and $\|q_j\|_{H^s(\Omega)}\leqslant M$, $j=1,2$, then there exists $\delta'\in(0,1)$ such that

$$\|q_1-q_2\|_{L^\infty(\Omega)}\leqslant C\big(\big\|\Lambda_{q_1}^\sharp-\Lambda_{q_2}^\sharp\big\|^{\delta'}+\big|\log\big(\big\|\Lambda_{q_1}^\sharp-\Lambda_{q_2}^\sharp\big\|\big)\big|^{-\delta'}\big).\tag{1.6}$$

As an immediate consequence of Theorem 1.1, we have the following uniqueness result.

Corollary 1.1. *Let $d \geqslant 2$ and q_j in $\mathcal{C}^{0,\mu}(\overline{\Omega})$, $j = 1, 2$. Assume $q_1 = q_2$ in a neighbourhood of the boundary Γ. Then there exists sufficiently large T such that $\Lambda^\sharp_{q_1} = \Lambda^\sharp_{q_2}$ implies $q_1 = q_2$ in Ω.*

In order to study the stability for the multidimensional Borg–Levinson theorem, we need to restrict the operator $\Lambda^\sharp_q$ to the following subspace

$$\mathcal{H}_1 = \left\{h \in H^{2d+4}\left(0, T; H^{3/2}(\Gamma)\right);\ \partial_t^j h(0, \cdot) = 0,\ 0 \leqslant j \leqslant 2d+3\right\}.$$

This subspace is endowed with the topology of $H^{2d+4}(0, T; H^{3/2}(\Gamma))$. By $\tilde{\Lambda}^\sharp_q$ we denote this restriction of $\Lambda^\sharp_q$, which defines a bounded operator from $\mathcal{H}_1$ into

$$\mathcal{H}_2 = L^2\left(0, T; H^s(\Gamma_0)\right)$$

for any $s \in [0, 1/2]$.

We have the following variant of Theorem 1.1, where $\|\cdot\|_s$ denotes the norm in $\mathcal{B}(\mathcal{H}_1; \mathcal{H}_2)$, the Banach space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$.

Theorem 1.2. *There exist $C > 0$, $\delta \in (0, 1)$ and sufficiently large T such that*

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leqslant C\left(\left\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\right\|_s^\delta + \left|\log\left(\left\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\right\|_s\right)\right|^{-\delta}\right) \tag{1.7}$$

for any $q_1, q_2 \in \mathfrak{X}(M, \omega)$.

If in addition $q_1, q_2 \in H^\alpha(\Omega)$, for $\alpha > d/2$ and $\|q_j\|_{H^\alpha(\Omega)} \leqslant M$, $j = 1, 2$, then there exists $\delta' \in (0, 1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\left(\left\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\right\|_s^{\delta'} + \left|\log\left(\left\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\right\|_s\right)\right|^{-\delta'}\right). \tag{1.8}$$

Now, for $0 \leqslant q \in L^\infty(\Omega)$, we denote by A_q the operator $A_q = -\Delta + q$ with the domain $\mathscr{D}(A_q) = H^1_0(\Omega) \cap H^2(\Omega)$. It is well known that the spectrum of A_q consists of a sequence of eigenvalues, counted according to their multiplicities:

$$0 \leqslant \lambda_{1,q} \leqslant \lambda_{2,q} \leqslant \cdots \leqslant \lambda_{k,q} \to +\infty.$$

The corresponding sequence of eigenfunctions is denoted by $(\varphi_{k,q})$. We may assume that this sequence forms an orthonormal basis of $L^2(\Omega)$.

In the sequel C denotes a generic positive constant depending only on Ω, Γ_0, ω and M. Since $\varphi_{k,q}$ is the solution of the following BVP

$$\begin{cases} (-\Delta + q)\varphi = \lambda_{k,q}\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma, \end{cases}$$

from the classical $H^2(\Omega)$ a priori estimates we derive

$$\|\varphi_{k,q}\|_{H^2(\Omega)} \leqslant C\lambda_{k,q}\|\varphi_{k,q}\|_{L^2(\Omega)} = C\lambda_{k,q}. \tag{1.9}$$

Therefore

$$\|\partial_\nu \varphi_{k,q}\|_{H^{1/2}(\Gamma)} \leqslant C\lambda_{k,q}.$$

On the other hand, there exists a positive constant $K \geqslant 1$ such that

$$K^{-1}k^{2/d} \leqslant \lambda_{k,q} \leqslant Kk^{2/d}. \tag{1.10}$$

Here K can be chosen uniformly in q with $0 \leqslant q(x) \leqslant M$ for $x \in \Omega$ (e.g., [9]). Therefore we have

$$\|\partial_\nu \varphi_{k,q}\|_{H^{1/2}(\Gamma_0)} \leqslant C\|\partial_\nu \varphi_{k,q}\|_{H^{1/2}(\Gamma)} \leqslant Ck^{2/d}.$$

We fix ζ such that $d/2+1 < \zeta \leqslant d+1$. Therefore it follows that

$$\left(k^{-2\zeta/d}\|\partial_\nu \varphi_{k,q}\|_{H^{1/2}(\Gamma_0)}\right) \in \ell^1.$$

We recall that ℓ^1 is the usual Banach space of real-valued sequences such that the corresponding series is absolutely convergent. This space is equipped with its natural norm.

Let $\mathbf{r} = (r_k)$ be the sequence given by $r_k = k^{-2\zeta/d}$ for each $k \geqslant 1$. We then introduce the following Banach space

$$\ell^1\left(H^{1/2}(\Gamma_0); \mathbf{r}\right) = \left\{g = (g_k);\ g_k \in H^{1/2}(\Gamma_0),\ k \geqslant 1,\ \text{and } \left(r_k\|g_k\|_{H^{1/2}(\Gamma_0)}\right) \in \ell^1\right\}.$$

The natural norm on this space is

$$\|g\|_{\ell^1(H^{1/2}(\Gamma_0);\mathbf{r})} = \sum_{k \geqslant 1} r_k\|g_k\|_{H^{1/2}(\Gamma_0)}.$$

Let $\mu = (\mu_k)$ be the sequence of eigenvalues of A_0 (that is A_q with $q = 0$). As a consequence of the min–max formula (e.g., [11]), we have

$$|\lambda_{k,q} - \mu_k| \leqslant \|q\|_{L^\infty(\Omega)}, \quad k \geqslant 1.$$

In other words, $\lambda_q = (\lambda_{k,q})$ belongs to the affine space $\tilde{\ell}^\infty = \mu + \ell^\infty$, where ℓ^∞ is the usual Banach space of bounded real-valued sequences. We equip $\tilde{\ell}^\infty$ with the following distance

$$d_\infty(\lambda_1, \lambda_2) = \left\|(\lambda_1 - \mu) - (\lambda_2 - \mu)\right\|_{\ell^\infty} = \|\lambda_1 - \lambda_2\|_{\ell^\infty},$$

if $\lambda_j \in \tilde{\ell}^\infty$, $j = 1, 2$.

We will apply Theorem 1.2 to prove the following one.

Theorem 1.3. *There exist $C > 0$ and $\mu_0 \in (0,1)$ such that the following estimate holds*

$$\|q_1 - q_2\|_{H^{-1/2}(\Omega)} \leqslant C\left(\left|\log\left(|\log \eta|\right)\right|\right)^{-\mu_0} \tag{1.11}$$

for any $q_1, q_2 \in \mathfrak{X}(M, \omega)$, where

$$\eta = d_\infty(\lambda_{q_1}, \lambda_{q_2}) + \|\partial_\nu \varphi_{q_1} - \partial_\nu \varphi_{q_2}\|_{\ell^1(H^{1/2}(\Gamma_0);\mathbf{r})}$$

is assumed to be small and $\partial_\nu \varphi_{q_j} = (\partial_\nu \varphi_{k,q_j})$, $j = 1, 2$.

If in addition $q_1, q_2 \in H^s(\Omega)$, for $s > d/2$ and $\|q_j\|_{H^s(\Omega)} \leqslant M$, $j = 1, 2$, then there exists $\mu_0' \in (0,1)$ such that

$$\|q_1 - q_2\|_{L^\infty(\Omega)} \leqslant C\left(\left|\log\left(|\log \eta|\right)\right|\right)^{-\mu_0'}. \tag{1.12}$$

Theorems 1.1 and 1.2 yield a single log-type stability estimate in the case where Neumann data are observed on any fixed subboundary $\Gamma_0 \times (0, T)$, provided that unknown coefficients q_j are given in a neighbourhood of Γ. The hyperbolic inverse problem consisting in the determination of the potential q in Eq. (1.1) from the full DN map defined by (1.2) was initiated by Rakesh and Symes [22]. They prove that the potential q can be recovered uniquely from the DN map Λ_q provided that the length of the time interval is larger than the diameter of the space domain Ω, see also [7] and [12]. A sharp uniqueness result was proved in [15] by the so-called boundary control method. We also refer to [3] and [14]. In the case of a piecewise constant potential q, Rakesh [21] proves the uniqueness and a stability estimate from the values of $\Lambda_q(f)$ where f is suitably chosen in a finite dimensional subspace. When the DN map is given on the whole lateral boundary Σ, the papers [10] and [25] establish Hölder stability estimates. Most recently, Bao and Yun [2] improve the result of [25]. Precisely, they prove a nearly Lipschitz stability estimate. The uniqueness and a Hölder stability estimate in a subdomain were established by Isakov and Sun [13] by a partial DN map. By the same method, Bellassoued, Jellali and Yamamoto [6] prove a log–log-type stability estimate in the case where Neumann data are observed on some large portion of Γ. Based on a method using global Carleman estimates, Bellassoued, Jellali and Yamamoto [5] prove a Lipschitz stability estimate when the potential is restricted to a finite dimensional subspace. The present work continues the one in [6] and generalizes it.

Theorem 1.3 is an extension of a result in [8] which is itself a variant of a theorem in [1]. To our knowledge, [1] is the only result which we found in the literature on stability estimates for multidimensional inverse spectral problems.

2. Preliminaries

This section is devoted to some estimates which are necessary in our analysis.

Let $u = u(t,x) \in H^2(Q)$ satisfy

$$u(0,\cdot) = \partial_t u(0,\cdot) = 0 \quad \text{in } \Omega, \qquad u(t,\cdot) = 0 \quad \text{on } \Sigma \tag{2.1}$$

and let $v = v(t,x) \in H^2(Q)$ be the solution to the following backward wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + q(x)v = 0 & \text{in } Q, \\ v(T,\cdot) = 0, \quad \partial_t v(T,\cdot) = 0 & \text{in } \Omega, \\ v = h & \text{on } \Sigma. \end{cases} \tag{2.2}$$

We have the following identity

$$\begin{aligned} \int_Q (\partial_t^2 - \Delta + q) u v \, dx \, dt &= \int_Q u (\partial_t^2 - \Delta + q) v \, dx \, dt - \int_\Sigma (v \partial_\nu u - u \partial_\nu v) \, dS \, dt \\ &= - \int_\Sigma v \partial_\nu u \, dS \, dt. \end{aligned} \tag{2.3}$$

This identity is seen by integration by parts and an application of Green's formula with respect to the variable x.

We recall a result on the existence of geometric optics solutions, which is due to Rakesh and Symes [22].

Lemma 2.1. *Let $\Phi \in C_0^\infty(\mathbb{R}^d)$, $\theta \in \mathbb{S}^{d-1}$, $\sigma > 0$ be arbitrarily given. Then the equation*

$$\partial_t^2 u - \Delta u + q(x) u = 0 \quad \text{in } Q$$

has a solution $u \in H^2(Q)$ of the form

$$u(t,x) = \Phi(x+t\theta)e^{i\sigma(x\cdot\theta+t)} + \psi_q(t,x;\sigma). \tag{2.4}$$

Here $\psi_q(t,x;\sigma)$ satisfies

$$\psi_q(t,x;\sigma) = 0, \quad (t,x) \in \Sigma,$$
$$\psi_q(s,x;\sigma) = \partial_t\psi_q(s,x;\sigma) = 0, \quad x \in \Omega,\ s = 0 \text{ or } T,$$

and

$$\sigma \left\|\psi_q(\cdot,\cdot;\sigma)\right\|_{L^2(Q)} + \left\|\nabla\psi_q(\cdot,\cdot;\sigma)\right\|_{L^2(Q)} \leqslant C\|\Phi\|_{H^3(\mathbb{R}^d)}, \tag{2.5}$$

where the constant C depends only on T, Ω and M and does not depend on Φ and σ.

We refer to [22] for the proof. We note that the H^2 regularity of $\psi_q(\cdot,\cdot;\sigma)$ can be derived from Theorem 2.1 in [19].

The following lemma shows a stability estimate in the continuation of the solutions of a hyperbolic equation from lateral boundary data on an arbitrary non-empty relatively open subset Γ_0 of Γ. The corresponding uniqueness in the continuation was proved by Robbiano [23].

We shall use the following notations. We choose $\varrho > 0$ such that

$$\omega(8\varrho) := \left\{x \in \Omega;\ \operatorname{dist}(x,\Gamma) \leqslant 8\varrho\right\} \subset \omega \tag{2.6}$$

and, for $\tau > 0$, we set

$$\omega_\tau = (0,\tau)\times\omega, \qquad \omega_\tau(\varrho) = (0,\tau)\times\omega(\varrho). \tag{2.7}$$

Lemma 2.2. *Let $q_1 \in \mathfrak{X}(M,\omega)$ and T be sufficiently large such that $T/3 > \operatorname{Diam}(\Omega)$. Let $w \in H^2(Q)$ be a solution of the following boundary value problem*

$$\begin{cases} \left(\partial_t^2 - \Delta + q_1(x)\right)w = F & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \end{cases} \tag{2.8}$$

where $F \in L^2(Q)$. Then there exist positive constants C, $T_1 > T/3$, μ and γ_0 such that the following estimate holds

$$\|w\|_{H^1(\omega_{T_1}(2\varrho))} \leqslant \frac{C}{\sqrt{\gamma}}\|w\|_{H^2(Q)} + e^{\mu\gamma}\left(\|F\|_{L^2(\omega_T)} + \|\partial_\nu w\|_{L^2(\Sigma_0)}\right) \tag{2.9}$$

for any $\gamma > \gamma_0$. Here the constant C depends on Ω, ω, T, M and is independent of w and q_1.

The proof of the lemma is given in Section 5.

3. Proof of Theorems 1.1 and 1.2

As in the previous works, our proof is essentially based on geometric optics solutions and the X-ray transform.

Without loss of generality, we can assume that $0 \in \Omega$. Moreover we assume that $T/3 > \operatorname{Diam}(\Omega)$. Let $\varepsilon > 0$ and $T_1 > 0$ satisfy $T_1 > \frac{T}{3}$ and $T_1 - 2\varepsilon > \operatorname{Diam}(\Omega)$. We set

$$\Omega_\varepsilon = \left\{x \in \mathbb{R}^d \setminus \overline{\Omega};\ \operatorname{dist}(x,\Omega) < \varepsilon\right\}. \tag{3.1}$$

We shall need a stability estimate for the problem of recovering a function from its X-ray transform. To $\Phi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$ and $\theta \in \mathbb{S}^{d-1}$ we associate $\widetilde{\Phi}_\theta$ defined by

$$\widetilde{\Phi}_\theta(x,t) = \Phi(x+t\theta), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R},$$

where Φ was extended by 0 outside Ω_ε.

Lemma 3.1. *Let $q_1, q_2 \in \mathfrak{X}(M,\omega)$, and let q be equal to $q_1 - q_2$ extended by 0 outside Ω. There exist sufficiently large $T_1 > 0$, $A > 0$ and $C > 0$ such that for any $\theta \in \mathbb{S}^{d-1}$ and any $\Phi \in \mathcal{C}_0^\infty(\Omega_\varepsilon)$*

$$\left| \int_0^{T_1} \int_\Omega \Phi^2(x) q(x - s\theta)\, dx\, ds \right| \leqslant C \left(\frac{1}{\gamma^{1/4}} + e^{A\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| \right) \|\Phi\|^2_{H^3(\mathbb{R}^d)}$$

for sufficiently large $\gamma > 0$. Here the constant C depends on Ω, ω, T, M and is independent of w and q_j.

Proof. It follows from Lemma 2.1 that if σ is sufficiently large, then the initial value problem

$$\left(\partial_t^2 - \Delta + q_2(x)\right) u = 0 \quad \text{in } Q_1 := (0,T_1) \times \Omega, \qquad u(0,\cdot) = \partial_t u(0,\cdot) = 0 \quad \text{in } \Omega$$

has a solution u_2 of the form

$$u_2(t,x) = \Phi(x+t\theta) e^{i\sigma(x\cdot\theta + t)} + \psi_{q_2}(t,x;\sigma), \tag{3.2}$$

where ψ_{q_2} satisfies

$$\psi_{q_2}(0,x;\sigma) = \partial_t \psi_{q_2}(0,x;\sigma) = 0, \quad x \in \Omega, \qquad \psi_{q_2}(t,x;\sigma) = 0 \quad \text{on } \Sigma_1 := (0,T_1) \times \Gamma \tag{3.3}$$

and

$$\sigma \left\| \psi_{q_2}(\cdot,\cdot;\sigma) \right\|_{L^2(Q_1)} + \left\| \nabla \psi_{q_2}(\cdot,\cdot;\sigma) \right\|_{L^2(Q_1)} \leqslant C \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{3.4}$$

Let $f_\sigma := u_{2|\Sigma_1} = \Phi(x+t\theta) e^{i\sigma(x\cdot\theta+t)}$ and let u_1 be the solution of the IBVP:

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + q_1(x) u_1 = 0 & \text{in } Q_1, \\ u_1(0,x) = \partial_t u_1(0,x) = 0 & \text{in } \Omega, \\ u_1 = u_2 := f_\sigma & \text{on } \Sigma_1. \end{cases}$$

We set $w = u_1 - u_2$ and $q(x) = q_2(x) - q_1(x)$. Then we easily prove

$$\begin{cases} \partial_t^2 w - \Delta w + q_1(x) w = q(x) u_2 & \text{in } Q_1, \\ w(0,x) = \partial_t w(0,x) = 0 & \text{in } \Omega, \\ w = 0 & \text{in } \Sigma_1. \end{cases}$$

We introduce a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{R}^d)$ satisfying $0 \leqslant \chi \leqslant 1$ and

$$\chi(x) = \begin{cases} 0, & x \in \omega(\varrho), \\ 1, & x \in \Omega \setminus \omega(2\varrho). \end{cases}$$

If $w_0(t,x) = \chi(x)w(t,x)$, then we see that w_0 is the solution of the following IBVP

$$\begin{cases} \partial_t^2 w_0 - \Delta w_0 + q_1(x) w_0 = q(x) u_2 - [\Delta, \chi] w & \text{in } Q_1, \\ w_0(0,\cdot) = \partial_t w_0(0,\cdot) = 0 & \text{in } \Omega, \\ w_0 = 0 & \text{on } \Sigma_1, \end{cases}$$

where we used $\chi(x)q(x) = q(x)$ in Ω because $q(x) = 0$ in ω.

On the other hand, for sufficiently large σ, Lemma 2.1 guarantees the existence of exponentially growing solutions v to the wave equation

$$\big(\partial_t^2 - \Delta + q_1(x)\big) v = 0 \quad \text{in } Q_1,$$

of the form

$$v(t,x) = \Phi(x + t\theta) e^{-i\sigma(x\cdot\theta + t)} + \psi_{q_1}(t,x;\sigma), \tag{3.5}$$

where ψ_{q_1} satisfies

$$\psi_{q_1}(t,x;\sigma) = 0, \quad (t,x) \in \Sigma_1,$$

$$\psi_{q_1}(T_1,x;\sigma) = \partial_t \psi_{q_1}(T_1,x;\sigma) = 0, \quad x \in \Omega,$$

and

$$\sigma \big\| \psi_{q_1}(\cdot,\cdot;\sigma) \big\|_{L^2(Q_1)} + \big\| \nabla \psi_{q_1}(\cdot,\cdot;\sigma) \big\|_{L^2(Q_1)} \leqslant C \|\Phi\|_{H^3(\mathbb{R}^d)}. \tag{3.6}$$

By $T_1 > \operatorname{diam}(\Omega) + 2\varepsilon$ and $\Phi \in C_0^\infty(\Omega_\varepsilon)$, we see that $\Phi(x + T_1\theta) = |\nabla\Phi(x + T_1\theta)| = 0$ in Ω. Hence we have

$$v(T_1,\cdot) = \partial_t v(T_1,\cdot) = 0 \quad \text{in } \Omega.$$

Then, by Green's formula, we obtain

$$\begin{aligned} \int_{Q_1} q(x) u_2(t,x) v \, dx\, dt - \int_{Q_1} [\Delta, \chi] w v \, dt\, dx &= \int_{Q_1} \big(\big(\partial_t^2 - \Delta + q_1(x)\big) w_0 \big) v \, dx\, dt \\ &= \int_{Q_1} w_0 \big(\partial_t^2 - \Delta + q_1(x)\big) v \, dx\, dt = 0 \end{aligned} \tag{3.7}$$

for an arbitrary $v \in H^1(Q)$.

It follows from (3.2), (3.5) and (3.7)

$$\begin{aligned} &\int_{Q_1} q(x) \Phi^2(x + t\theta) \, dx\, dt + \int_{Q_1} q(x) \Phi(x + t\theta) \big(\psi_{q_1}(t,x;\sigma) e^{i\sigma(x\cdot\theta + t)} + \psi_{q_2}(t,x;\sigma) e^{-i\sigma(x\cdot\theta + t)} \big) \, dx\, dt \\ &\quad + \int_{Q_1} q(x) \psi_{q_1}(t,x;\sigma) \psi_{q_2}(t,x;\sigma) \, dx\, dt \\ &= \int_{Q_1} [\Delta, \chi] w(t,x) v(t,x) \, dt\, dx. \end{aligned} \tag{3.8}$$

Now (3.4) and (3.6) imply

$$\begin{aligned}&\left|\int_{Q_1} q(x)\Phi(x+t\theta)\big(\psi_{q_1}(t,x;\sigma)e^{i\sigma(x\cdot\theta+t)}+\psi_{q_2}(t,x;\sigma)e^{-i\sigma(x\cdot\theta+t)}\big)\,dx\,dt\right|\\&\quad\leqslant \frac{C}{|\sigma|}\|\Phi\|^2_{H^3(\mathbb{R}^d)}\end{aligned}$$

and

$$\left|\int_{Q_1} q(x)\psi_{q_1}(t,x;\sigma)\psi_{q_2}(t,x;\sigma)\,dx\,dt\right|\leqslant \frac{C}{\sigma^2}\|\Phi\|^2_{H^3(\mathbb{R}^d)}.$$

Furthermore

$$\begin{aligned}\left|\int_{Q_1}[\Delta,\chi]w(t,x)v(t,x)\,dx\,dt\right| &\leqslant C\|w\|_{H^1(\omega_{T_1}(2\varrho))}\|v\|_{L^2(Q_1)}\\&\leqslant C\|\Phi\|_{H^3(\mathbb{R}^d)}\|w\|_{H^1(\omega_{T_1}(2\varrho))}.\end{aligned}\tag{3.9}$$

Hence, by (3.8) we obtain

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\,dx\,dt\right|\leqslant \frac{C}{\sigma}\|\Phi\|^2_{H^3(\mathbb{R}^d)}+C\|w\|_{H^1(\omega_{T_1}(2\varrho))}\|\Phi\|_{H^3(\mathbb{R}^d)}.$$

This and Lemma 2.2 yield

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\,dx\,dt\right|\leqslant \frac{C\|\Phi\|^2_{H^3(\mathbb{R}^d)}}{\sigma}+C\left(\frac{\|w\|_{H^2(Q)}}{\sqrt{\gamma}}+e^{A\gamma}\|\partial_\nu w\|_{L^2(\Sigma_0)}\right)\|\Phi\|_{H^3(\mathbb{R}^d)}.\tag{3.10}$$

Here we used $F=q(x)u_2\equiv 0$ in ω. By (3.4), the energy estimate for the equation in w and $f_\sigma=\Phi(x+t\theta)e^{i\sigma(x\cdot\theta+t)}$ on Σ_1, we have

$$\|w\|_{H^2(Q)}\leqslant C\sigma\|\Phi\|_{H^3(\mathbb{R}^d)}\tag{3.11}$$

and

$$\begin{aligned}\|\partial_\nu w\|_{L^2(\Sigma_0)}=\left\|\Lambda^\sharp_{q_1}(f_\sigma)-\Lambda^\sharp_{q_2}(f_\sigma)\right\|_{L^2(\Sigma_0)}&\leqslant C\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\|f_\sigma\|_{H^{1,1}(\Sigma)}\\&\leqslant C\sigma^2\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\|\Phi\|_{H^3(\mathbb{R}^d)}.\end{aligned}\tag{3.12}$$

Now (3.10), (3.11) and (3.12) imply

$$\left|\int_{Q_1} q(x)\Phi^2(x+t\theta)\,dx\,dt\right|\leqslant C\left(\frac{1}{\sigma}+\frac{\sigma}{\sqrt{\gamma}}+\sigma^2e^{A\gamma}\left\|\Lambda^\sharp_{q_1}-\Lambda^\sharp_{q_2}\right\|\right)\|\Phi\|^2_{H^3(\mathbb{R}^d)}.\tag{3.13}$$

Choosing $\sigma = \gamma^{1/4}$, we find

$$\left| \int_0^{T_1} \int_\Omega q(x)\Phi^2(x+t\theta)\,dx\,dt \right| \leqslant C\left(\frac{1}{\gamma^{1/4}} + e^{A\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| \right) \|\Phi\|_{H^3(\mathbb{R}^d)}^2.$$

By using the substitution $x \to x + s\theta$, we obtain the desired estimate. $\square$

Lemma 3.2. *There exist four constants $C > 0$, $A > 0$, $\delta > 0$ and $\gamma_0 > 0$ such that*

$$\left| \int_{\mathbb{R}} q(y+s\theta)\,ds \right| \leqslant \frac{C}{\gamma^\delta} + Ce^{A\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\|, \quad \text{a.e. } y \in \mathbb{R}^d,$$

for any $\gamma \geqslant \gamma_0$ and all $\theta \in \mathbb{S}^{d-1}$.

Proof. We fix $\theta \in \mathbb{S}^{d-1}$ and a positive function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ which is supported in the unit ball and satisfies $\|\varphi\|_{L^2(\mathbb{R}^d)} = 1$. We define

$$\Phi_\kappa(x) = \kappa^{-d/2}\varphi\left(\frac{x-y}{\kappa}\right),$$

where $y \in \Omega_\varepsilon$ and $\kappa > 0$ is small enough.

If

$$h(x,\theta) = \int_0^{T_1} q(x-t\theta)\,dt,$$

then

$$\begin{aligned} \left|h(y,\theta)\right| &= \left| \int_{\mathbb{R}^d} \Phi_\kappa^2(x)h(y,\theta)\,dx \right| \\ &\leqslant \left| \int_{\mathbb{R}^d} \Phi_\kappa^2(x)h(x,\theta)\,dx \right| + \left| \int_{\mathbb{R}^d} \Phi_\kappa^2(x)\big(h(y,\theta)-h(x,\theta)\big)\,dx \right|. \end{aligned}$$

Since

$$\left|h(y,\theta)-h(x,\theta)\right| \leqslant \begin{cases} C|x-y|^\mu & \text{if } q_j \in \mathcal{C}^{0,\mu}(\mathbb{R}^d), \\ C|x-y|^{\mu'} & \text{if } q_j \in H^s(\mathbb{R}^d), \end{cases}$$

where $\mu' = s - \frac{d}{2}$ if $s - \frac{d}{2} < 1$ and μ' is an arbitrary number such that $0 \leqslant \mu' < 1$ if $s - \frac{d}{2} \geqslant 1$, taking $\kappa > 0$ sufficiently small and applying Lemma 3.1, we obtain

$$\left|h(y,\theta)\right| \leqslant C\left(\frac{1}{\gamma^{1/4}} + e^{\mu\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| \right) \|\Phi_\kappa\|_{H^3(\mathbb{R}^d)}^2 + C\int_{\mathbb{R}^d} \big(|x-y|^\mu + |x-y|^{\mu'}\big)\Phi_\kappa^2(x)\,dx.$$

On the other hand,

$$\|\Phi_\kappa\|_{L^2(\mathbb{R}^d)} = 1, \qquad \|\Phi_\kappa\|_{H^3(\mathbb{R}^d)} \leqslant C\kappa^{-3},$$

and, for $\mu_0 = \min(\mu, \mu')$,

$$\int_{\mathbb{R}^d} \left(|x-y|^{\mu} + |x-y|^{\mu'}\right) \Phi_\kappa^2(x)\,dx \leqslant C\kappa^{\mu_0}.$$

Then by Lemma 3.1 we have, for all $\theta \in \mathbb{S}^{d-1}$,

$$\left| \int_0^{T_1} q(y-t\theta)\,dt \right| \leqslant \frac{C}{\gamma^{1/4}}\kappa^{-6} + C\kappa^{-6} e^{\mu\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| + C\kappa^{\mu_0}, \quad \text{a.e. } y \in \Omega_\varepsilon.$$

We select κ satisfying

$$\kappa^{\mu_0} = \frac{1}{\gamma^{1/4}}\kappa^{-6}.$$

From the last estimate, there exist $\delta > 0$ and $B > 0$ such that

$$\left| \int_{-T_1}^{T_1} q(y+t\theta)\,dt \right| \leqslant \frac{C}{\gamma^{\delta}} + Ce^{B\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\|, \quad \text{a.e. } y \in \Omega_\varepsilon.$$

Using that $T_1 > \mathrm{Diam}(\Omega)$ and $\mathrm{supp}(q) \subset \Omega$, we obtain

$$\left| \int_{\mathbb{R}} q(y+t\theta)\,dt \right| \leqslant \frac{C}{\gamma^{\delta}} + Ce^{B\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\|, \quad \text{a.e. } y \in \mathbb{R}^d,$$

for all $\theta \in \mathbb{S}^{d-1}$.

The proof is then completed. □

We set

$$\mathcal{P}(q)(\theta, x) = \int_{\mathbb{R}} q(x+t\theta)\,dt, \quad x \in \mathbb{R}^d,\ \theta \in \mathbb{S}^{d-1}.$$

Proof of Theorem 1.1. We identify q with its zero extension outside Ω. From Lemma 3.2 we have

$$\left|\mathcal{P}(q)(x,\theta)\right| = \left| \int_{\mathbb{R}} q(x+s\theta)\,ds \right| \leqslant C\left(\frac{1}{\gamma^{\delta}} + e^{\mu\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| \right), \quad \text{a.e. } x \in \mathbb{R}^d.$$

We choose $R > 0$ such that $\Omega \subset B(0,R)$. Then

$$\begin{aligned} \left\|\mathcal{P}(q)\right\|_{L^2(\mathcal{T})}^2 &:= \int_{\mathbb{S}^{d-1}} \int_{\theta^\perp} \left|\mathcal{P}(q)(\theta,y)\right|^2 dy\,d\theta \\ &= \int_{\mathbb{S}^{d-1}} \int_{\theta^\perp \cap B(0,R)} \left|\mathcal{P}(q)(\theta,y)\right|^2 dy\,d\theta \leqslant C\left(\frac{1}{\gamma^{\delta}} + e^{\mu\gamma} \left\| \Lambda_{q_1}^\sharp - \Lambda_{q_2}^\sharp \right\| \right), \end{aligned}$$

where $\mathcal{T} = \{(\theta, y);\ \theta \in \mathbb{S}^{d-1},\ y \in \theta^\perp\}$ is the tangent bundle.

We recall the following well-known estimate for the X-ray transform (e.g., Theorem 5.1, p. 42 in [20]):

$$\|q\|_{H^{-\frac{1}{2}}(\Omega)} \leqslant C\|\mathcal{P}(q)\|^2_{L^2(\mathcal{T})}.$$

Therefore

$$\|q\|_{H^{-\frac{1}{2}}(\Omega)} \leqslant C\left(\frac{1}{\gamma^\delta} + e^{\mu\gamma}\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right). \tag{3.14}$$

The estimate above is valid if $\gamma \geqslant \gamma_0$. Hence there exists a sufficiently small $\epsilon_0 > 0$ such that if $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| < \epsilon_0$ and

$$\gamma = \frac{1-\delta}{\mu}\left|\log\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right|,$$

then we have $\gamma \geqslant \gamma_0$. From (3.14) we obtain

$$\|q\|_{H^{-1/2}(\Omega)} \leqslant C\left(\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^\delta + C'\left|\log\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|\right|^{-\delta}\right) \tag{3.15}$$

if $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| < \epsilon_0$. On the other hand, $\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\| \geqslant \epsilon_0$ directly implies

$$\|q\|_{H^{-1/2}(\Omega)} \leqslant C\|q\|_{L^\infty(\Omega)} \leqslant \frac{2CM}{\epsilon_0^\delta}\epsilon_0^\delta \leqslant C'\|\Lambda^\sharp_{q_1} - \Lambda^\sharp_{q_2}\|^\delta.$$

Thus (3.15) holds for all $q_1, q_2 \in \mathfrak{X}(M,\omega)$. That is, the proof of (1.5) is completed.

The second estimate is a consequence of the Sobolev imbedding theorem and an interpolation inequality. Let $\eta > 0$ be such that $s = d/2 + 2\eta$. Then we have

$$\|q\|_{L^\infty(\Omega)} \leqslant C\|q\|_{H^{s-\eta}(\Omega)} \leqslant C\|q\|^\alpha_{H^{-1/2}(\Omega)}\|q\|^{1-\alpha}_{H^s(\Omega)} \leqslant C\|q\|^\alpha_{H^{-1/2}(\Omega)},$$

where

$$\alpha = \frac{s-d/2}{2s+1} < 1.$$

The conclusion (1.6) follows from (3.15). □

Proof of Theorem 1.2. We use the same notations as in Lemma 3.1. From (3.12) we deduce

$$\begin{aligned}
\|\partial_\nu w\|_{L^2(\Sigma_0)} &= \|\tilde{\Lambda}^\sharp_{q_1}(f_\sigma) - \tilde{\Lambda}^\sharp_{q_2}(f_\sigma)\|_{L^2(\Sigma_0)} \\
&\leqslant C\|\tilde{\Lambda}^\sharp_{q_1}(f_\sigma) - \tilde{\Lambda}^\sharp_{q_2}(f_\sigma)\|_{L^2(0,T;H^s(\Gamma_0))} \\
&\leqslant C\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\|_s\|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\Gamma))}.
\end{aligned}$$

On the other hand, one can easily establish the following estimate (a proof will be given in [8]):

$$\|f_\sigma\|_{H^{2d+4}(0,T;H^{3/2}(\Gamma))} \leqslant C\sigma^{2d+4}\|\Phi\|_{H^{6+2d}(\mathbb{R}^d)}.$$

Therefore

$$\|\partial_\nu w\|_{L^2(\Sigma_0)} \leqslant C\sigma^{2d+4}\big\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\big\|_s.$$

In this case, in place of (3.13) we have

$$\left|\int_Q q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\sigma} + \frac{\sigma}{\sqrt{\gamma}} + \sigma^{2d+4}e^{\mu\gamma}\big\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\big\|_s\right)\|\Phi\|^2_{H^{6+2d}(\mathbb{R}^d)}.$$

As in Lemma 3.1, by taking $\sigma = \gamma^{1/4}$, we find

$$\left|\int_Q q(x)\Phi^2(x+t\theta)\,dx\,dt\right| \leqslant C\left(\frac{1}{\gamma^{1/4}} + e^{C\gamma}\big\|\tilde{\Lambda}^\sharp_{q_1} - \tilde{\Lambda}^\sharp_{q_2}\big\|_s\right)\|\Phi\|^2_{H^{6+2d}(\mathbb{R}^d)}.$$

That is, we have Lemma 3.1 with $\Lambda^\sharp_{q_i}$ replaced by $\tilde{\Lambda}^\sharp_{q_i}$. The rest of the proof is similar to that of Theorem 1.1. $\square$

4. Proof of Theorem 1.3

We first define an elliptic DN map. Let $q \in L^\infty(\Omega)$, $\sigma(A_q) = \{\lambda_{k,q}\}$ be the spectrum of A_q and $\rho(A_q) = \mathbb{C} \setminus \sigma(A_q)$ be the resolvent set of A_q. From well-known results (e.g., [18]), for any $\lambda \in \rho(A_q)$ and $f \in H^{3/2}(\Gamma)$, the nonhomogeneous boundary value problem

$$\begin{cases} -\Delta u + qu - \lambda u = 0 & \text{in } \Omega,\\ u = f & \text{on } \Gamma \end{cases}$$

has a unique solution $u_{q,f} \in H^2(\Omega)$ and the DN map

$$\Pi^\sharp_q(\lambda) : f \to \partial_\nu u_{q,f}|_{\Gamma_0}$$

defines a bounded operator from $H^{3/2}(\Gamma)$ into $H^{1/2}(\Gamma_0)$.

We recall

$$\mathcal{H}_1 = \big\{h \in H^{2d+4}\big(0,T;H^{\frac{3}{2}}(\Gamma)\big);\ \partial_t^j h(0,\cdot) = 0,\ 0 \leqslant j \leqslant 2d+3\big\},$$

and let

$$\mathscr{R}^\sharp_q h = \sum_{k\geqslant 1}\frac{1}{\lambda^{d+2}_{q,k}}(\partial_\nu\varphi_{q,k})|\Gamma_0\int_0^t \frac{\sin\sqrt{\lambda_{q,k}}(t-s)}{\sqrt{\lambda_{q,k}}}\big\langle -\partial_s^{2(d+2)}h(\cdot,s), \partial_\nu\varphi_{q,k}\big\rangle\,ds,$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Gamma)$-scalar product. Then $\mathscr{R}^\sharp_q$ defines a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2 = L^2(0,T;H^s(\Gamma_0))$.

We shall need in the sequel the following three lemmas. Their proof can be found in [1] or can be deduced easily from the results in this reference (see also [8]).

We fix $0 \leqslant s \leqslant \frac{1}{2}$.

Lemma 4.1. *Let $q \in L^\infty(\Omega)$. Then for any $m > \frac{d}{2}$, $f \in H^{3/2}(\Gamma)$ and $\lambda \in \rho(A_q)$, we have*

$$\frac{d^m}{d\lambda^m}\Pi_q^\sharp(\lambda)f = -m!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q}-\lambda)^{m+1}}\langle f, \partial_\nu\varphi_{k,q}\rangle\partial_\nu\varphi_{k,q}|_{\Gamma_0}.$$

Lemma 4.2. *Let N be a nonnegative integer and let $q_1, q_2 \in L^\infty(\Omega)$ satisfy $0 \leqslant q_1, q_2 \leqslant M$ for some positive constant M. Then there exists a positive constant C, depending only on Ω and M, such that*

$$\left\| \frac{d^p}{d\lambda^p}\left[\Pi_{q_1}^\sharp(\lambda) - \Pi_{q_2}^\sharp(\lambda)\right]\right\|_s \leqslant \frac{C}{|\lambda|^{p+\frac{1-2s}{4}}}, \quad \lambda \leqslant 0 \text{ and } 0 \leqslant p \leqslant N,$$

where $\|\cdot\|_s$ denotes the norm in $\mathcal{L}(H^{3/2}(\Gamma);\ H^s(\Gamma_0))$.

Lemma 4.3. *For each $h \in \mathcal{H}_1$, we have*

$$\tilde{\Lambda}_q^\sharp h = \sum_{j=0}^{d+1}\left[\frac{d^j}{d\lambda^j}\Pi_q^\sharp(\lambda)\right]_{|\lambda=0}\left(-\partial_t^2 h\right) + \mathscr{R}_q^\sharp h, \tag{4.1}$$

where $\tilde{\Lambda}_q^\sharp$ is as in Theorem 1.2.

By this lemma, for $q \in L^\infty(\Omega)$ we note that $\tilde{\Lambda}_q^\sharp$ defines a bounded operator from $\mathcal{H}_1$ into $\mathcal{H}_2 = L^2(0, T; H^s(\Gamma_0))$.

Lemma 4.4. *Let $q_1, q_2 \in \mathfrak{X}(M, \omega)$. There exist $C > 0$, $\delta \in (0, 1)$ and $\gamma_0 > 0$ such that the following estimate holds*

$$\begin{aligned} &C\left\|\partial_\nu(\varphi_{k,q_1} - \varphi_{k,q_2})\right\|_{L^2(\Gamma)} \\ &\quad \leqslant (\lambda_{k,q_1} + \lambda_{k,q_2})\gamma^{-\delta} + e^{A\gamma}\left(\left\|\partial_\nu(\varphi_{k,q_1} - \varphi_{k,q_2})\right\|_{L^2(\Gamma_0)} + |\lambda_{k,q_1} - \lambda_{k,q_2}|\right) \end{aligned} \tag{4.2}$$

for any $\gamma \geqslant \gamma_0$ and any $k \geqslant 1$. Here the constants C, δ and γ_0 are independent of q_j.

Proof. We introduce the function $\psi_k(x) = (\varphi_{k,q_1} - \varphi_{k,q_2})(x)$. We easily see that ψ_k satisfies the boundary value problem

$$\begin{cases} (-\Delta + q_1 - \lambda_{k,q_1})\psi_k = (q_2 - q_1)\varphi_{k,q_2} + (\lambda_{k,q_1} - \lambda_{k,q_2})\varphi_{k,q_2} & \text{in } \Omega, \\ \psi_k = 0 & \text{on } \Gamma. \end{cases}$$

We fix $T > 0$ sufficiently large and we consider the function

$$w_k(t, x) = e^{it\sqrt{\lambda_{k,q_1}}}\psi_k(x), \quad t \in (0, T). \tag{4.3}$$

Then w_k solves the following initial value problem

$$\begin{cases} \left(\partial_t^2 - \Delta + q_1(x)\right)w_k = F_k & \text{in } Q, \\ w_k(t, x) = 0 & \text{on } \Sigma, \end{cases} \tag{4.4}$$

where

$$F_k = e^{it\sqrt{\lambda_{k,q_1}}}\left((q_2 - q_1)\varphi_{k,q_2} + (\lambda_{k,q_1} - \lambda_{k,q_2})\varphi_{k,q_2}\right).$$

Since $q_2 - q_1 = 0$ in ω, we obtain

$$\|F_k\|_{L^2(\omega_T)} \leqslant C|\lambda_{k,q_1} - \lambda_{k,q_2}|. \tag{4.5}$$

Applying Lemma 2.2 and using (4.5), we obtain

$$\|w_k\|_{H^1(\omega_{T_1}(2\varrho))} \leqslant \frac{C}{\sqrt{\gamma}}\|w_k\|_{H^2(Q)} + e^{\mu\gamma}\big(|\lambda_{k,q_1} - \lambda_{k,q_2}| + \|\partial_\nu w_k\|_{L^2(\Sigma_0)}\big). \tag{4.6}$$

The last estimate implies

$$\begin{aligned}&\|\varphi_{k,q_1} - \varphi_{k,q_2}\|_{H^1(\omega(2\varrho))}\\ &\quad\leqslant \frac{C}{\sqrt{\gamma}}(\lambda_{k,q_1} + \lambda_{k,q_2}) + e^{\mu\gamma}\big(|\lambda_{k,q_1} - \lambda_{k,q_2}| + \big\|\partial_\nu(\varphi_{k,q_1} - \varphi_{k,q_2})\big\|_{L^2(\Sigma_0)}\big).\end{aligned} \tag{4.7}$$

On the other hand, using (1.9) and a classical interpolation inequality, we have

$$\begin{aligned}\big\|\partial_\nu(\varphi_{k,q_1} - \varphi_{k,q_2})\big\|_{L^2(\Gamma)} &\leqslant C\|\varphi_{k,q_1} - \varphi_{k,q_2}\|_{H^{3/2}(\omega(2\varrho))}\\ &\leqslant C\|\varphi_{k,q_1} - \varphi_{k,q_2}\|^{1/2}_{H^1(\omega(2\varrho))}\|\varphi_{k,q_1} - \varphi_{k,q_2}\|^{1/2}_{H^2(\omega(2\varrho))}\\ &\leqslant \gamma^{1/4}\|\varphi_{k,q_1} - \varphi_{k,q_2}\|_{H^1(\omega(2\varrho))} + \gamma^{-1/4}(\lambda_{k,q_1} + \lambda_{k,q_2}).\end{aligned} \tag{4.8}$$

Combining (4.8) and (4.7), we obtain (4.2). □

We set now $Z(\lambda) = (\Pi^\sharp_{q_1}(\lambda) - \Pi^\sharp_{q_2}(\lambda))$. Then from Taylor's formula, for $1 \leqslant j \leqslant d$, we derive

$$Z^{(j)}(0) = \sum_{p=j}^{d} \frac{(-\lambda)^{p-j}}{(p-j)!} Z^{(p)}(\lambda) + \int_\lambda^0 \frac{(-\tau)^{d-j}}{(d-j)!} Z^{(d+1)}(\tau)\, d\tau. \tag{4.9}$$

Lemma 4.5. *There exist $C > 0$ and $\mu_1 \in (0,1)$ such that the following estimate*

$$\big\|Z^{(d+1)}(\lambda)\big\|_s \leqslant C|\log\eta|^{-\mu_1} \tag{4.10}$$

holds true for any $\lambda \leqslant 0$. Here C is a positive constant depending on M, Ω and ω and $\|\cdot\|_s$ denotes the norm in $\mathcal{L}(H^{3/2}(\Gamma); H^s(\Gamma_0))$.

Proof. Let $f \in H^{3/2}(\Gamma)$. It follows from Lemma 4.1

$$\begin{aligned}Z^{(d+1)}(\lambda)f = &-(d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_1} - \lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_1}\rangle\partial_\nu\varphi_{k,q_1}|_{\Gamma_0}\\ &+ (d+1)!\sum_{k\geqslant 1}\frac{1}{(\lambda_{k,q_2} - \lambda)^{d+2}}\langle f, \partial_\nu\varphi_{k,q_2}\rangle\partial_\nu\varphi_{k,q_2}|_{\Gamma_0}.\end{aligned}$$

We split $Z^{(d+1)}(\lambda)f$ into three terms $Z^{(d+1)}(\lambda)f = \mathcal{I}_1(\lambda) + \mathcal{I}_2(\lambda) + \mathcal{I}_3(\lambda)$, where

$$\mathcal{I}_1(\lambda) = -(d+1)! \sum_{k\geqslant 1} \left[\frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}} \right] \langle f, \partial_\nu \varphi_{k,q_1} \rangle \partial_\nu \varphi_{k,q_1} |_{\Gamma_0},$$

$$\mathcal{I}_2(\lambda) = -(d+1)! \sum_{k\geqslant 1} \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}} \langle f, \partial_\nu \varphi_{k,q_1} - \partial_\nu \varphi_{k,q_2} \rangle \partial_\nu \varphi_{k,q_1} |_{\Gamma_0},$$

$$\mathcal{I}_3(\lambda) = -(d+1)! \sum_{k\geqslant 1} \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}} \langle f, \partial_\nu \varphi_{k,q_2} \rangle [\partial_\nu \varphi_{k,q_1} - \partial_\nu \varphi_{k,q_2}] |_{\Gamma_0}.$$

For $\mathcal{I}_1$, we have

$$\left\| \mathcal{I}_1(\lambda) \right\|_{H^{1/2}(\Gamma)} \leqslant (d+1)! \|f\|_{L^2(\Gamma)} \sum_{k\geqslant 1} \left| \frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}} \right| \|\partial_\nu \varphi_{k,q_2}\|^2_{H^{1/2}(\Gamma)}.$$

On the other hand, noting $\lambda \leqslant 0$, $\lambda_{k,q_j} \geqslant 0$, $j = 1, 2$, we see that

$$\begin{aligned} \left| \frac{1}{(\lambda_{k,q_1}-\lambda)^{d+2}} - \frac{1}{(\lambda_{k,q_2}-\lambda)^{d+2}} \right| &\leqslant C \max\left(\frac{1}{\lambda_{k,q_1}^{d+3}}, \frac{1}{\lambda_{k,q_2}^{d+3}} \right) |\lambda_{k,q_1} - \lambda_{k,q_2}| \\ &\leqslant \frac{C}{k^{\frac{2(d+3)}{d}}} |\lambda_{k,q_1} - \lambda_{k,q_2}|, \end{aligned}$$

where we used estimate (1.10). On the other hand, since (see (1.9) and (1.10))

$$\|\partial_\nu \varphi_{k,q_2}\|^2_{H^{1/2}(\Gamma)} \leqslant C k^{\frac{4}{d}},$$

we obtain

$$\begin{aligned} \left\| \mathcal{I}_1(\lambda) \right\|_{H^{1/2}(\Gamma)} &\leqslant C \|f\|_{L^2(\Gamma)} d_\infty(\lambda_{q_1}, \lambda_{q_2}) \sum_{k\geqslant 1} \frac{1}{k^{\frac{2(d+1)}{d}}} \\ &\leqslant C \|f\|_{L^2(\Gamma)} d_\infty(\lambda_{q_1}, \lambda_{q_2}) \\ &\leqslant C\eta \|f\|_{L^2(\Gamma)}. \end{aligned} \tag{4.11}$$

For $\mathcal{I}_2(\lambda)$, we have

$$\left\| \mathcal{I}_2(\lambda) \right\|_{H^{1/2}(\Gamma)} \leqslant C \|f\|_{L^2(\Gamma)} \sum_{k\geqslant 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \left\| \partial_\nu (\varphi_{k,q_1} - \varphi_{k,q_2}) \right\|_{L^2(\Gamma)}. \tag{4.12}$$

Then Lemma 4.4 yields

$$\begin{aligned} \sum_{k\geqslant 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \left\| \partial_\nu (\varphi_{k,q_1} - \varphi_{k,q_2}) \right\|_{L^2(\Gamma)} &\leqslant C\gamma^{-\delta} \sum_{k\geqslant 1} \frac{\lambda_{k,q_1}(\lambda_{k,q_1} + \lambda_{k,q_2})}{(\lambda_{k,q_2}-\lambda)^{d+2}} \\ &\quad + e^{A\gamma} \sum_{k\geqslant 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} \left\| \partial_\nu (\varphi_{k,q_1} - \varphi_{k,q_2}) \right\|_{L^2(\Gamma_0)} \\ &\quad + e^{A\gamma} \sum_{k\geqslant 1} \frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}} |\lambda_{k,q_1} - \lambda_{k,q_2}|. \end{aligned} \tag{4.13}$$

Therefore, using (1.10), we obtain

$$\sum_{k\geqslant 1}\frac{\lambda_{k,q_1}}{(\lambda_{k,q_2}-\lambda)^{d+2}}\left\|\partial_\nu(\varphi_{k,q_1}-\varphi_{k,q_2})\right\|_{L^2(\Gamma)}\leqslant C\gamma^{-\delta}+e^{A\gamma}\eta. \tag{4.14}$$

Minimizing with respect to γ, we find

$$\left\|\mathcal{I}_2(\lambda)\right\|_{H^{1/2}(\Gamma)}\leqslant C\|f\|_{L^2(\Gamma)}|\log\eta|^{-\mu_1} \tag{4.15}$$

with some $\mu_1\in(0,1)$.

For $\mathcal{I}_3(\lambda)$, we have

$$\begin{aligned}\left\|\mathcal{I}_3(\lambda)\right\|_{H^{1/2}(\Gamma)}&\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\frac{1}{\lambda_{k,q_2}^{d+1}}\|\partial_\nu\varphi_{k,q_1}-\partial_\nu\varphi_{k,q_2}\|_{H^{1/2}(\Gamma_0)}\\&\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\frac{1}{k^{\frac{2(d+1)}{d}}}\|\partial_\nu\varphi_{k,q_2}-\partial_\nu\varphi_{k,q_1}\|_{H^{1/2}(\Gamma_0)}\\&\leqslant C\|f\|_{L^2(\Gamma)}\sum_{k\geqslant 1}\frac{1}{k^{\frac{2\zeta}{d}}}\|\partial_\nu\varphi_{k,q_2}-\partial_\nu\varphi_{k,q_1}\|_{H^{1/2}(\Gamma_0)}.\end{aligned}$$

Therefore

$$\|\mathcal{I}_3\|_{H^{1/2}(\Gamma)}\leqslant C\eta\|f\|_{L^2(\Gamma)}. \tag{4.16}$$

The conclusion follows then from a combination of (4.16), (4.15) and (4.11). □

Proof of Theorem 1.3. From (4.9) and Lemma 4.2, we obtain

$$\left\|Z^{(j)}(0)\right\|_s\leqslant C\left(|\lambda|^{-j-\frac{1-2s}{4}}+|\lambda|^{d-j+1}|\log\eta|^{-\mu_1}\right),$$

and then

$$\left\|Z^{(j)}(0)\right\|_s\leqslant C\left(|\lambda|^{-\frac{1-2s}{4}}+|\lambda|^{d+1}|\log\eta|^{-\mu_1}\right),\quad \text{if } |\lambda|\geqslant 1.$$

In particular

$$\left\|Z^{(j)}(0)\right\|_s\leqslant C\min_{\rho\geqslant 1}\left(\rho^{-\frac{1-2s}{4}}+\rho^{d+1}|\log\eta|^{-\mu_1}\right)=C|\log\eta|^{-\mu_2}, \tag{4.17}$$

where $\mu_2\in(0,1)$.

Let $\mathscr{R}_q^\sharp$ be defined as in Lemma 4.3. We can proceed as in the proof of Lemma 4.5 to prove

$$\left\|\mathscr{R}_{q_1}^\sharp-\mathscr{R}_{q_2}^\sharp\right\|_s\leqslant C|\log\eta|^{-\mu_3}. \tag{4.18}$$

From (4.1), (4.18) and (4.17), we deduce

$$\left\|\tilde{\Lambda}_{q_1}^\sharp-\tilde{\Lambda}_{q_2}^\sharp\right\|_s\leqslant C|\log\eta|^{-\mu_4},$$

provided that η is sufficiently small.

This and estimates in Theorem 1.2 lead to the desired result. □

5. Proof of Lemma 2.2

We prove Lemma 2.2. We first remark that, changing T by $2T$ and shifting the time variable, we are reduced to the case $Q = (-T, T) \times \Omega$, $\Sigma = (-T, T) \times \Gamma$ and

$$\omega_\tau = (-\tau, \tau) \times \omega, \qquad \omega_\tau(\rho) = (-\tau, \tau) \times \omega(\rho).$$

In the sequel $w \in H^2(Q)$ denotes a solution of the following initial value problem:

$$\begin{cases} \partial_t^2 w - \Delta w + q_1(x) w = F(t, x) & \text{in } Q, \\ w(x, t) = 0 & \text{on } \Sigma, \end{cases} \tag{5.1}$$

where $F \in L^2(Q)$.

5.1. Preliminary and elliptic estimation

This section contains preliminary materials, needed to prove the conclusion (2.9) for the solutions of (5.1). We set

$$\omega(\varrho_1, \rho_2) = \left\{x \in \Omega;\ \varrho_1 \leqslant \operatorname{dist}(x, \Gamma) \leqslant \varrho_2\right\} \subset \Omega, \quad \varrho_1 < \varrho_2 < 8\varrho,$$

and, for $r > 0$,

$$\begin{aligned} &\Omega_r = (-r, r) \times \Omega, \qquad \omega_r(\varrho, 3\varrho) = (-r, r) \times \omega(\varrho, 3\varrho), \\ &\Gamma_r = (-r, r) \times \Gamma, \qquad \Sigma_{0,r} = (-r, r) \times \Gamma_0. \end{aligned} \tag{5.2}$$

Let $\theta \in \mathcal{C}_0^\infty(\mathbb{R})$ be a cut-off function defined by

$$\theta(t) = \begin{cases} 1, & |t| \leqslant (T-2), \\ 0, & |t| \geqslant (T-1). \end{cases} \tag{5.3}$$

We introduce the partial Fourier–Bros–Iagolnitzer (F.B.I.) transformation $\mathcal{F}_\gamma$. It is defined for $u \in \mathcal{S}(\mathbb{R}^{n+1})$, the space of rapidly decreasing functions, by

$$u_{\gamma,t}(s, x) = \mathcal{F}_\gamma u(z, x) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2} \theta(y) u(y, x)\, dy, \quad z = t + is. \tag{5.4}$$

We recall the following well-known estimate (see [24])

$$\left|\partial_x^\alpha \mathcal{F}_\gamma u(z, x)\right| \leqslant C \sqrt{\frac{\gamma}{2\pi}} e^{\lambda s^2} e^{-\frac{\gamma}{2}[\operatorname{dist}(t, \operatorname{supp}(\theta u))]^2} \sup_{x \in \mathbb{R}^n} \left\|\partial_x^\alpha u(\cdot, x)\right\|_{L^2(\mathbb{R})} \tag{5.5}$$

for any $u \in \mathcal{C}_0^\infty(\mathbb{R} \times \mathbb{R}^n)$.

Next, we assume that T is sufficiently large, $s \in [-3r, 3r]$ and $t \in [-\frac{T}{2}, \frac{T}{2}]$. We choose a cut-off function χ satisfying $0 \leqslant \chi \leqslant 1$, $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, and

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \omega(6\varrho), \\ 0 & \text{if } x \in \Omega \setminus \omega(7\varrho). \end{cases} \tag{5.6}$$

Let $w(t,x)$ be a solution to (5.1). Setting $u(t,x)=\chi(x)w(t,x)$, we obtain

$$\begin{cases} \partial_t^2 u-\Delta u+q_1(x)u=[\Delta,\chi]w+\chi(x)F(t,x) & \text{in } Q=(-T,T)\times\Omega, \\ u(t,x)=0 & \text{in } \Sigma=(-T,T)\times\Gamma. \end{cases} \tag{5.7}$$

In connection with the operator $\partial_t^2-\Delta+q_1(x)$, we define an elliptic operator by

$$P(x,D_{x,s})=\partial_s^2+\Delta_x-q_1(x). \tag{5.8}$$

Since

$$\partial_s\int_{\mathbb{R}} e^{-\frac{\gamma}{2}(is+t-y)^2}\theta(y)u(y,x)\,dy=i\int_{\mathbb{R}} e^{-\frac{\gamma}{2}(z-y)^2}\partial_y\big[\theta(y)u(y,x)\big]\,dy, \tag{5.9}$$

we have

$$\begin{gathered} Pu_{\gamma,t}(s,x)=R_{\gamma,t}(s,x)+G_{\gamma,t}(s,x),\quad (s,x)\in\Omega_{3r}, \\ u_{\gamma,t}(s,x)=0,\quad (s,x)\in\ \Sigma_{3r}, \end{gathered} \tag{5.10}$$

where

$$R_{\gamma,t}(s,x)=-\sqrt{\frac{\gamma}{2\pi}}\int e^{-\frac{\gamma}{2}(z-y)^2}\big(2\theta'(y)\partial_t u(y,x)+\theta''(y)u(y,x)\big)\,dy \tag{5.11}$$

and

$$G_{\gamma,t}(s,x)=\sqrt{\frac{\gamma}{2\pi}}\int e^{-\frac{\gamma}{2}(z-t)^2}\theta(y)\big([\Delta,\chi]w(y,x)+\chi(x)F(y,x)\big)\,dy. \tag{5.12}$$

As θ' and θ'' are supported in $|y|\geqslant(T-2)$, there exists $\eta>0$, independent of T, such that

$$\|R_{\gamma,t}\|_{L^2(\Omega_{3r})}\leqslant Ce^{-\eta\gamma T}\|u\|_{H^1(Q)},\quad \forall t\in\left[-\frac{T}{2},\frac{T}{2}\right]. \tag{5.13}$$

Moreover there exists $\alpha>0$, independent of T, such that

$$\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\leqslant Ce^{\alpha\gamma}\|u\|_{H^1(Q)},\quad \forall t\in\left[-\frac{T}{2},\frac{T}{2}\right]. \tag{5.14}$$

By (5.12) and (5.6) we easily obtain

$$\|G_{\gamma,t}\|_{L^2(\omega_{3r}(6\varrho))}=\|F_{\gamma,t}\|_{L^2(\omega_{3r}(6\varrho))}\leqslant Ce^{\alpha\gamma}\|F\|_{L^2(\omega_T(7\varrho))}, \tag{5.15}$$

where $F_{\gamma,t}(s,x)=\mathcal{F}_\gamma F(z,x)$.

Let K be a compact subset contained in $(-3r,3r)\times\overline{\Omega}$ and $\psi(s,x)$ be a $\mathcal{C}^1$-function satisfying $\nabla_{s,x}\psi(s,x)\neq 0$ on K. Let

$$\varphi(s,x)=e^{-\beta\psi(s,x)}, \tag{5.16}$$

where $\beta > 0$ is sufficiently large. Then the following Carleman estimate holds (see [17] for the proof): there exists $\tau_0 > 0$ such that

$$C\tau \|e^{\tau\varphi}u\|^2_{H^1_\tau(\Omega_{3r})} \leqslant \|e^{\tau\varphi}Pu\|^2_{L^2(\Omega_{3r})} + \tau\|e^{\tau\varphi}u\|^2_{H^1_\tau(\Sigma_{3r})}, \tag{5.17}$$

whenever $u \in \mathcal{C}_0^\infty(K)$ and $\tau > \tau_0$.
Here and henceforth we set

$$\|u\|^2_{H^1_\tau(\Omega_{3r})} = \|\nabla_{s,x}u\|^2_{L^2(\Omega_{3r})} + \tau^2\|u\|^2_{L^2(\Omega_{3r})} \tag{5.18}$$

and

$$\|u\|^2_{H^1_\tau(\Sigma_{3r})} = \|u\|^2_{H^1(\Sigma_{3r})} + \tau^2\|u\|^2_{L^2(\Sigma_{3r})}. \tag{5.19}$$

We further introduce a cut-off function $\mathcal{X}$ satisfying $0 \leqslant \mathcal{X} \leqslant 1$, $\mathcal{X} \in \mathcal{C}^\infty(\mathbb{R})$, and

$$\mathcal{X}(\rho) = \begin{cases} 0 & \text{if } \rho \leqslant \frac{1}{4},\ \rho \geqslant 9, \\ 1 & \text{if } \rho \in [\frac{1}{2}, 8]. \end{cases} \tag{5.20}$$

Now we proceed to the estimation near Γ_0.

5.2. Estimation near the boundary part Γ_0

We shall begin to estimate $u_{\gamma,t}$ in a ball $B_1 = B(x^{(1)}, r) = \{x \in \mathbb{R}^n;\ |x - x^{(1)}| < r\}$ over a small interval $(-r, r)$ by the velocity trace (in the normal direction) in the given part $\Sigma_{0,3r} = (-3r, 3r) \times \Gamma_0 \subset \Sigma_{3r}$.

Lemma 5.1. *Let $u_{\gamma,t}$ be a solution to* (5.10)*. Then there exist $B_1^* \equiv (-r, r) \times B_1 \subset \Omega_{3r}$ and $\nu_0 \in (0, 1)$ such that the following estimate holds*:

$$\|u_{\gamma,t}\|_{H^1(B_1^*)} \leqslant C\big(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))} + \|\partial_\nu u_{\lambda,t}\|_{L^2(\Sigma_{0,3r})}\big)^{\nu_0}\big(\|u_{\lambda,t}\|_{H^1(\Omega_{3r})}\big)^{1-\nu_0} \tag{5.21}$$

for some positive constant C.

Proof. Let us choose $r > 0$ small and $x^{(0)} \in \mathbb{R}^n \setminus \overline{\Omega}$ such that

$$r < \frac{\varrho}{4}, \qquad \overline{B(x^{(0)}, r)} \cap \overline{\Omega} = \emptyset, \qquad B(x^{(0)}, 2r) \cap \Omega \neq \emptyset, \qquad B(x^{(0)}, 4r) \cap \Gamma \subset \Gamma_0. \tag{5.22}$$

That is, $x^{(0)}$ is an outer point of $\overline{\Omega}$ and is near Γ_0. We define the functions $\psi_0(s, x)$ and $\varphi_0(s, x)$ by

$$\psi_0(s, x) = |x - x^{(0)}|^2 + s^2, \qquad \varphi_0(s, x) = e^{-\frac{\beta}{r^2}\psi_0(s,x)}. \tag{5.23}$$

Denote

$$v_{\gamma,t}(s, x) = \mathcal{X}\left(\frac{\psi_0}{r^2}\right)u_{\gamma,t}(s, x). \tag{5.24}$$

Taking into account that $u_{\gamma,t} = 0$ on Γ and applying Carleman estimate (5.17), we obtain

$$C\tau\|e^{\tau\varphi_0}v_{\gamma,t}\|^2_{H^1_\tau(\Omega_{3r})} \leqslant \|e^{\tau\varphi_0}Pv_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \tau\left\|e^{\tau\varphi_0}\partial_\nu\left(\mathcal{X}\left(\frac{\psi_0}{r^2}\right)u_{\gamma,t}\right)\right\|^2_{L^2(\Sigma_{3r})} \tag{5.25}$$

for $\tau > \tau_0$. Therefore by (5.24) and (5.15), we have

$$\begin{aligned} P v_{\gamma,t}(s,x) &= \mathcal{X}\left(\frac{\psi_0}{r^2}\right) P u_{\gamma,t}(s,x) + \left[P, \mathcal{X}\left(\frac{\psi_0}{r^2}\right)\right] u_{\gamma,t}(s,x) \\ &= \mathcal{X}\left(\frac{\psi_0}{r^2}\right)\big(R_{\gamma,t}(s,x) + G_{\gamma,t}(s,x)\big) + \left[P, \mathcal{X}\left(\frac{\psi_0}{r^2}\right)\right] u_{\gamma,t}(s,x) \\ &= \mathcal{X}\left(\frac{\psi_0}{r^2}\right)\big(R_{\gamma,t}(s,x) + \chi(x) F_{\gamma,t}(s,x)\big) + \left[P, \mathcal{X}\left(\frac{\psi_0}{r^2}\right)\right] u_{\gamma,t}(s,x). \end{aligned} \tag{5.26}$$

Since $[P, \mathcal{X}(\frac{\psi_0}{r^2})]$ is supported in

$$\left|x - x^{(0)}\right|^2 + s^2 \leqslant \frac{1}{2} r^2, \qquad 8r^2 \leqslant \left|x - x^{(0)}\right|^2 + s^2 \leqslant 9r^2, \tag{5.27}$$

we see, in view of (5.22), that $|x - x^{(0)}| \geqslant r$ for all $x \in \overline{\Omega}$ and $\Omega \cap \{x;\ |x - x^{(0)}|^2 \leqslant \frac{1}{2} r^2\} = \emptyset$. Therefore

$$\begin{aligned} C\tau e^{2\tau e^{-7\beta}} \|u_{\gamma,t}\|^2_{H^1_\tau((r^2 \leqslant \psi_0 \leqslant 7r^2) \cap \Omega_{3r})} &\leqslant e^{2\tau e^{-8\beta}} \|u_{\gamma,t}\|^2_{H^1(\psi_0 \leqslant 9r^2)} + e^{2\tau e^{-\beta}} \|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} \\ &\quad + e^{2\tau e^{-\beta}} \|\chi F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\varrho))} + \tau e^{2\tau e^{-\beta}} \|\partial_\nu u_{\gamma,t}\|^2_{L^2(\Sigma_{0,3r})}. \end{aligned} \tag{5.28}$$

We can now select $r > 0$ and $x^{(1)} \in \Omega$ such that

$$\operatorname{dist}\big(x^{(1)}, \Gamma\big) \geqslant 4r, \qquad B_1^* = (-r, r) \times B\big(x^{(1)}, r\big) \subset \big\{r^2 \leqslant \psi_0(s,x) \leqslant 7r^2\big\}. \tag{5.29}$$

Then for $\tau > \tau_0$ we have

$$\begin{aligned} \|u_{\gamma,t}\|^2_{H^1(B_1^*)} &\leqslant C e^{C_1 \tau} \big[\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\varrho))} + \|\partial_\nu u_{\gamma,t}\|^2_{L^2(\Sigma_0,3r)}\big] \\ &\quad + e^{-C_2 \tau} \|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})}. \end{aligned} \tag{5.30}$$

Minimizing the right-hand side with respect to γ and setting $\nu_0 = \frac{C_2}{C_1 + C_2}$, we obtain the following estimate

$$\|u_{\gamma,t}\|^2_{H^1(B_1^*)} \leqslant \big(\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\varrho))} + \|\partial_\nu u_{\gamma,t}\|^2_{H^1(\Sigma_{0,3r})}\big)^{\nu_0} \big(\|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})}\big)^{1-\nu_0}. \tag{5.31}$$

The restriction $\tau > \tau_0$ requires a separated treatment for the minimization only if

$$\|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})} \leqslant C\big[\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\varrho))} + \|\partial_\nu u_{\gamma,t}\|^2_{L^2(\Sigma_0,3r)}\big], \tag{5.32}$$

in which case (5.21) is trivial. This completes the proof of the lemma. □

5.3. Estimation in $\omega_r(\varrho, 3\varrho)$

In this subsection we extend the estimation from B_1^* to $\omega_r(\varrho, 3\varrho)$. To accomplish this, we use the techniques developed in [4]. This is done by continuing estimates (5.21). Let $B(x^{(j)}, r)$, $2 \leqslant j \leqslant N$, be a covering of $\omega(\varrho, 3\varrho)$. We can assume that $x^{(j)}$ satisfies $\operatorname{dist}(x^{(j)}, \Gamma) \geqslant 4r$. In the sequel, we assume without loss of generality that

$$B\big(x^{(j+1)}, r\big) \subset B\big(x^{(j)}, 2r\big), \tag{5.33}$$

and we set

$$B_j^* = (-r, r) \times B\big(x^{(j)}, r\big), \quad 2 \leqslant j \leqslant N.$$

Lemma 5.2. *Let $u_{\gamma,t}$ be a solution to* (5.10)*. Then there exist constants $\nu \in (0,1)$ and $C > 0$ such that the following estimate holds*:

$$\|u_{\gamma,t}\|_{H^1(B^*_{k+1})} \leqslant \big(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|^2_{L^2(\omega_{3r}(7\varrho))} + \|u_{\gamma,t}\|_{H^1(B_k^*)}\big)^{\nu}\big(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\big)^{1-\nu} \tag{5.34}$$

for any $k \geqslant 1$.

Proof. In order to prove (5.34), we define the functions $\psi_k(s,x)$ and $\varphi_k(s,x)$ by

$$\psi_k(s,x) = \big|x - x^{(k)}\big|^2 + s^2, \qquad \varphi_k(s,x) = e^{-\frac{\beta}{r^2}\psi_k(s,x)}. \tag{5.35}$$

Moreover we set

$$v_{\gamma,t}(s,x) = \mathcal{X}\left(\frac{\psi_k}{r^2}\right) u_{\gamma,t}(s,x).$$

By applying Carleman estimate (5.17) in the interior domain, we obtain

$$C\tau \big\|e^{\tau\varphi_k} v_{\gamma,t}\big\|^2_{H^1_\tau(\Omega_{3r})} \leqslant \big\|e^{\tau\varphi_k} P v_{\gamma,t}\big\|^2_{L^2(\Omega_{3r})}. \tag{5.36}$$

In the same way as above, we have

$$P v_{\gamma,t}(s,x) = \mathcal{X}\left(\frac{\psi_k}{r^2}\right)\big(R_{\gamma,t}(s,x) + \chi F_{\gamma,t}(s,x)\big) + \left[P, \mathcal{X}\left(\frac{\psi_k}{r^2}\right)\right] u_{\lambda,t}(s,x). \tag{5.37}$$

Since $[P, \mathcal{X}(\frac{\psi_k}{r^2})]$ is supported in

$$\frac{r^2}{4} \leqslant \big|x - x^{(0)}\big|^2 + s^2 \leqslant \frac{r^2}{2}, \qquad 8r^2 \leqslant \big|x - x^{(0)}\big|^2 + s^2 \leqslant 9r^2, \tag{5.38}$$

we combine with (5.36) and (5.37), so that

$$\begin{aligned} C\tau e^{2\tau e^{-7\beta}} \|u_{\gamma,t}\|^2_{H^1_\tau(r^2 \leqslant \psi_k \leqslant 7r^2)} &\leqslant e^{2\tau e^{-\beta/4}} \|u_{\gamma,t}\|^2_{H^1(\psi_k \leqslant r^2/2)} + e^{2\tau e^{-8\beta}} \|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})} \\ &\quad + e^{2\tau e^{-\beta/4}} \big(\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|^2_{L^2(\Omega_{3r})}\big), \end{aligned} \tag{5.39}$$

and hence

$$\begin{aligned} C e^{2\tau e^{-7\beta}} \|u_{\gamma,t}\|^2_{H^1_\tau(\psi_k \leqslant 7r^2)} &\leqslant e^{2\tau e^{-\beta/4}} \|u_{\gamma,t}\|^2_{H^1(\psi_k \leqslant r^2/2)} + e^{2\tau e^{-8\beta}} \|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})} \\ &\quad + e^{2\tau e^{-\beta/4}} \big(\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|^2_{L^2(\Omega_{3r})}\big). \end{aligned} \tag{5.40}$$

Thus we obtain

$$\begin{aligned} \|u_{\gamma,t}\|^2_{H^1(\psi_k \leqslant 7r^2)} &\leqslant e^{C_1\tau} \big[\|u_{\gamma,t}\|^2_{H^1(\psi_k \leqslant r^2/2)} + \|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|^2_{L^2(\Omega_{3r})}\big] \\ &\quad + e^{-C_2\tau} \|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})}. \end{aligned} \tag{5.41}$$

Now minimizing the right-hand side with respect to τ and setting $\nu = \frac{C_2}{C_1+C_2}$, we obtain the following estimate:

$$\|u_{\gamma,t}\|^2_{H^1(\psi_k\leqslant 7r^2)} \leqslant \left(\|R_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\gamma,t}\|^2_{H^1(\psi_k\leqslant r^2/2)}\right)^{\nu}\left(\|u_{\gamma,t}\|^2_{H^1(\Omega_{3r})}\right)^{1-\nu}. \quad (5.42)$$

Since

$$B^*_{k+1} \subset \left\{\psi_k(s,x) \leqslant 7r^2\right\}, \qquad \left\{\psi_k(x,s) \leqslant r^2/2\right\} \subset B^*_k, \quad (5.43)$$

we obtain (5.34). This completes the proof of the lemma. □

Lemma 5.3. *Let $u_{\gamma,t}$ be a solution to* (5.10)*. Then there exist constants $C > 0$ and $\mu = \nu^N$ such that the following estimate holds*

$$\|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \leqslant C\left(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|u_{\gamma,t}\|_{H^1(B^*_1)}\right)^{\mu}\left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\mu}. \quad (5.44)$$

Here $\nu \in (0,1)$ is the constant given in Lemma 5.2*.*

Proof. We set

$$\alpha_k = \|u_{\gamma,t}\|_{H^1(B^*_k)}, \qquad A = \|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|_{L^2(\Omega_{3r})}, \qquad B = \|u_{\lambda,t}\|_{H^1(\Omega_{3r})}. \quad (5.45)$$

Then by (5.34) we have

$$\alpha_{k+1} \leqslant B^{1-\nu}(\alpha_k + A)^{\nu}. \quad (5.46)$$

Applying Lemma 4 in [17], we obtain, for all $\mu \in (0, \nu^n]$,

$$\alpha_n \leqslant 2^{\frac{1}{1-\nu}} B^{1-\mu}(\alpha_1 + A)^{\mu}. \quad (5.47)$$

Therefore

$$\|u_{\gamma,t}\|_{H^1(B^*_n)} \leqslant C\left(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi F_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|u_{\gamma,t}\|_{H^1(B^*_1)}\right)^{\mu}\left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\mu} \quad (5.48)$$

for any $n = 2,\dots,N$ and $\mu \in (0,\nu^N]$. Patching estimates (5.48) for $n = 1,\dots,N$, we obtain (5.44).
This completes the proof of the lemma. □

Lemma 5.4. *Let $u_{\gamma,t}$ be a solution to* (5.10) *and $r_0 = r/2$. Then there exist constants $C > 0$ and $\kappa \in (0,1)$ such that the following estimate holds*:

$$\begin{aligned}&\|u_{\gamma,t}\|_{H^1(\omega_{r_0}(2\varrho))}\\&\quad\leqslant C\left(\|R_{\gamma,t}\|_{L^2(\Omega_r)} + \|F_{\gamma,t}\|_{L^2(\omega_r(7\varrho))} + \|u_{\gamma,t}\|_{H^1(\omega_r(2\varrho,3\varrho))}\right)^{\kappa}\left(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\right)^{1-\kappa}\end{aligned} \quad (5.49)$$

for any $t \in (-T/2, T/2)$.

Proof. We introduce a cut-off function χ_0 satisfying $0 \leqslant \chi_0 \leqslant 1$, $\chi_0 \in \mathcal{C}^\infty(\mathbb{R}^n)$ and

$$\chi_0(x) = \begin{cases} 1 & \text{if } x \in \omega(2\varrho),\\ 0 & \text{if } x \in \Omega \setminus \omega(3\varrho).\end{cases} \quad (5.50)$$

Let $u_{\gamma,t}(s,x)$ be a solution to (5.10). We set $\tilde{u}_{\gamma,t}(s,x) = \chi_0(x)u_{\gamma,t}(s,x)$.

We use a Carleman estimate (see inequality (1) of Section 3 in [17]) to obtain

$$\|\tilde{u}_{\gamma,t}\|_{H^1(\Omega_{r_0})} \leqslant \|\tilde{u}_{\gamma,t}\|^{\kappa}_{H^1(\Omega_r)} \big(\|P\tilde{u}_{\gamma,t}\|_{H^1(\Omega_r)} + \|\tilde{u}_{\gamma,t}\|_{H^1(\omega_r(2\varrho,3\varrho))}\big)^{1-\kappa}. \tag{5.51}$$

Proceeding as for (5.37), we have

$$\begin{aligned} P\tilde{u}_{\gamma,t}(x,s) &= \chi_0(x)Pu_{\gamma,t}(x,s) + [P,\chi_0]u_{\gamma,t}(x,s) \\ &= \chi_0\big(R_{\gamma,t}(x,s) + G_{\gamma,t}(x,s)\big) + [P,\chi_0]u_{\gamma,t}(x,s) \\ &= \chi_0\big(R_{\gamma,t}(x,s) + F_{\gamma,t}(x,s)\big) + [P,\chi_0]u_{\gamma,t}(x,s). \end{aligned} \tag{5.52}$$

Since $[P,\chi_0]$ is supported in $(-r,r)\times\omega(2\varrho,3\varrho)$, we have

$$\|\tilde{u}_{\gamma,t}\|_{H^1(\Omega_{r_0})} \leqslant \big(\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|\chi_0 F_{\gamma,t}\|^2_{L^2(\Omega_{3r})} + \|u_{\gamma,t}\|_{H^1(\omega_r(2\varrho,3\varrho))}\big)^{\kappa}\big(\|u_{\gamma,t}\|_{H^1(\Omega_{3r})}\big)^{1-\kappa}. \tag{5.53}$$

This completes the proof of the lemma. $\Box$

Lemma 5.5. *Let $u_{\gamma,t}$ be a solution to* (5.10)*. Then there exist $C>0$, $\alpha>0$ and sufficiently large $T>0$ such that the following estimate holds*:

$$C\|u_{\gamma,t}\|^2_{H^1(\omega_{r_0}(2\varrho))} \leqslant e^{-\alpha\gamma}\|u\|^2_{H^1(Q)} + e^{C\gamma}\big(\|\partial_\nu u\|^2_{L^2(\Sigma_0)} + \|F\|^2_{L^2(\Omega_T(7\varrho))}\big) \tag{5.54}$$

for all $t \in [-\frac{T}{2}, \frac{T}{2}]$.

Proof. From Lemma 5.3 and Young's inequality, we easily obtain

$$\|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \leqslant \epsilon^k\|u_{\gamma,t}\|_{H^1(\Omega_{3r})} + \epsilon^{-k'}\big[\|R_{\gamma,t}\|_{L^2(\Omega_{3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))} + \|u_{\gamma,t}\|_{H^1(B_1^*)}\big] \tag{5.55}$$

for all $\epsilon>0$. Here

$$k = \frac{1}{1-\mu}, \quad k' = \frac{1}{\mu}, \quad \text{and} \quad \mu = \nu^N. \tag{5.56}$$

Using estimates (5.13) and (5.14), we have, for all $t\in[-\frac{T}{2},\frac{T}{2}]$,

$$\begin{aligned} &\|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \\ &\quad\leqslant \epsilon^k e^{\alpha\gamma}\|u\|_{H^1(Q)} + \epsilon^{-k'}\big[e^{-\eta T\gamma}\|u\|_{H^1(Q)} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))} + \|u_{\gamma,t}\|_{H^1(B_1^*)}\big]. \end{aligned} \tag{5.57}$$

Selecting in (5.57)

$$\epsilon = e^{-\frac{2\alpha}{k}\gamma}$$

we find

$$\begin{aligned} \|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \leqslant{}& e^{-\alpha\gamma}\|u\|_{H^1(Q)} + e^{-(\eta T - \frac{2\alpha k'}{k})\gamma}\|u\|_{H^1(Q)} \\ &+ e^{\frac{2\alpha k'}{k}\gamma}\big(\|u_{\gamma,t}\|_{H^1(B_1^*)} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big) \end{aligned} \tag{5.58}$$

for all $t \in [-\frac{T}{2}, \frac{T}{2}]$ and $\gamma > 0$. Take T sufficiently large such that

$$\eta T - \frac{2\alpha k'}{k} > \alpha \tag{5.59}$$

and we obtain from (5.58)

$$\|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \leqslant e^{-\alpha\gamma}\|u\|_{H^1(Q)} + e^{\theta\gamma}\big(\|u_{\gamma,t}\|_{H^1(B_1^*)} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big), \tag{5.60}$$

where we set

$$\theta = \frac{2\alpha k'}{k}. \tag{5.61}$$

Similarly, we obtain from Lemma 5.1 and Young's inequality

$$\begin{aligned} &\|u_{\gamma,t}\|_{H^1(B_1^*)} \\ &\quad\leqslant \epsilon^{k_0}e^{\alpha\gamma}\|u\|_{H^1(Q)} + \epsilon^{-k_0'}\big[e^{-\eta T\gamma}\|u\|_{H^1(Q)} + \|\partial_\nu u_{\gamma,t}\|_{L^2(\Sigma_{0,3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big], \end{aligned} \tag{5.62}$$

where

$$k_0 = \frac{1}{1-\nu_0}, \qquad k_0' = \frac{1}{\nu_0}. \tag{5.63}$$

Selecting $\epsilon = e^{-(\frac{2\alpha+\theta}{k_0})\gamma}$, we deduce that, for some positive constant θ',

$$\begin{aligned} \|u_{\gamma,t}\|_{H^1(B_1^*)} &\leqslant e^{-(\alpha+\theta)\gamma}\|u\|_{H^1(Q)} + e^{-(\eta T - \frac{(2\alpha+\theta)k_0'}{k_0})\gamma}\|u\|_{H^1(Q)} \\ &\quad + e^{\theta'\gamma}\big(\|\partial_\nu u_{\gamma,t}\|_{L^2(\Sigma_{0,3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big). \end{aligned} \tag{5.64}$$

We assume that T is sufficiently large in such a way that

$$\left(\eta T - \frac{(2\alpha+\theta)k_0'}{k_0}\right) > (\alpha+\theta). \tag{5.65}$$

Then, by (5.64), we obtain

$$\|u_{\gamma,t}\|_{H^1(B_1^*)} \leqslant e^{-(\alpha+\theta)\gamma}\|u\|_{H^1(Q)} + e^{\theta'\gamma}\big(\|\partial_\nu u_{\gamma,t}\|_{L^2(\Sigma_{0,3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big). \tag{5.66}$$

Inserting (5.66) in (5.60), we obtain

$$\|u_{\gamma,t}\|_{H^1(\omega_r(\varrho,3\varrho))} \leqslant e^{-\alpha\gamma}\|u\|_{H^1(Q)} + e^{C_0\gamma}\big(\|\partial_\nu u_{\gamma,t}\|_{L^2(\Sigma_{0,3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big) \tag{5.67}$$

for some positive constant C_0.

By the same method we obtain by Lemma 5.4, (5.67) and Young's inequality

$$\begin{aligned} &\|u_{\gamma,t}\|_{H^1(\omega_{r_0}(2\varrho))} \\ &\quad\leqslant \epsilon^{k_1}e^{\alpha\gamma}\|u\|_{H^1(Q)} + \epsilon^{-k_1'}\big[e^{-\eta T\gamma}\|u\|_{H^1(Q)} + \|u_{\gamma,t}\|_{H^1(\omega_r(2\varrho,3\varrho))} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big], \end{aligned} \tag{5.68}$$

where

$$k_1 = \frac{1}{1-\kappa}, \qquad k_1' = \frac{1}{\kappa}. \tag{5.69}$$

Selecting $\epsilon = e^{-(\frac{2\alpha+C_0}{k_1})\gamma}$, we obtain for some positive constant θ'

$$\begin{aligned} \|u_{\gamma,t}\|_{H^1(\omega_{r_0}(2\varrho))} &\leqslant e^{-(\alpha+C_0)\gamma}\|u\|_{H^1(Q)} + e^{-(\eta T - \frac{(2\alpha+C_0)k_1'}{k_1})\gamma}\|u\|_{H^1(Q)} \\ &\quad + e^{C\gamma}\big(\|\partial_\nu u_{\gamma,t}\|_{L^2(\Sigma_{0,3r})} + \|F_{\gamma,t}\|_{L^2(\omega_{3r}(7\varrho))}\big). \end{aligned} \tag{5.70}$$

If $T > 0$ is sufficiently large such that

$$\eta T - \frac{(2\alpha+C_0)k_1'}{k_1} > \alpha + C_0, \tag{5.71}$$

then by (5.70) we obtain (5.54). $\square$

We are now ready to complete the proof of Lemma 2.2.

Lemma 5.6. *Let $u \in H^2(Q)$ be a solution of* (5.7)*. Let $T_1 = T/2 - r_0$. Then there exist $C > 0$ and $\mu > 0$ such that the following estimate holds*:

$$\|u\|_{H^1(\omega_{T_1}(2\varrho))} \leqslant \frac{C}{\sqrt{\gamma}}\|u\|_{H^2(\Omega)} + e^{\mu\gamma}\big(\|\partial_\nu u\|_{L^2(\Sigma_0)} + \|F\|_{L^2(\omega_T(7\varrho))}\big). \tag{5.72}$$

Proof. We set $u_\gamma(t,x) = u_{\gamma,t}(0,x)$. Then we have

$$u_\gamma(t,x) = \sqrt{\frac{\gamma}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\gamma}{2}(t-y)^2}\theta(y)u(y,x)\,dy = (K_\gamma * \theta u)(t,x),$$

where

$$K_\gamma(t) = \sqrt{\frac{\gamma}{2\pi}} e^{-\frac{\gamma}{2}t^2}.$$

By $\widehat{u}(\eta,x)$ we denote the Fourier transform of $u(t,x)$ with respect to t. We have

$$\widehat{\theta u}(\eta,x) - \widehat{u_\gamma}(\eta,x) = (1 - \widehat{K_\gamma})\widehat{\theta u}(\eta,x).$$

Furthermore we can directly verify that

$$\big|(1 - \widehat{K_\gamma})(\eta)\big| \leqslant \frac{\eta^2}{\gamma},$$

so that we obtain for $T_1 = T/2 - r_0$

$$\|u - u_\gamma\|_{L^2(\omega_{T_1}(2\varrho))} \leqslant \frac{C}{\sqrt{\gamma}}\|u\|_{H^1(Q)}.$$

Similarly we have

$$\|u-u_\gamma\|_{H^1(\omega_{T_1}(2\varrho))} \leqslant \frac{C}{\sqrt{\gamma}}\|u\|_{H^2(Q)}.$$

Hence

$$\begin{aligned}\|u\|_{H^1(\omega_{T_1}(2\varrho))} &\leqslant C\big[\|u-u_\gamma\|_{H^1(\omega_{T_1}(2\varrho))}+\|u_\gamma\|_{H^1(\omega_{T_1}(2\varrho))}\big]\\ &\leqslant C\bigg[\frac{1}{\sqrt{\gamma}}\|u\|_{H^2(Q)}+\|u_\gamma\|_{H^1(\omega_{T_1}(2\varrho))}\bigg].\end{aligned}$$

On the other hand, by the Cauchy inequality (see Appendix A in [4]), we obtain

$$\|u_\gamma\|^2_{H^1(\omega_{T_1}(2\varrho))} \leqslant e^{-\mu\gamma}\|u\|^2_{H^1(Q)} + e^{\mu'\gamma}\big(\|u\|^2_{L^2(\Sigma_0)} + \|F\|_{L^2(\omega_T(7\varrho))}\big)$$

for some positive constants μ and μ'. This completes the proof of the lemma. □

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Appendix A. An IBVP for the wave equation

Our aim in this appendix is to prove that the IBVP (1.1) has a unique sufficiently regular solution in order to insure that the operator defined in (1.2) is bounded.

We need to consider the following IBVP

$$\begin{cases}\partial_t^2 u-\Delta u=F & \text{in } Q,\\ u(0,\cdot)=u_0, \quad \partial_t u(0,\cdot)=u_1 & \text{in } \Omega,\\ u=f & \text{on } \Sigma.\end{cases} \tag{A.1}$$

We recall the following theorem, stated as Theorem 2.1 in Lasiecka, Lions and Triggiani [16].

Theorem A.1. *Let $F\in L^1(0,T;L^2(\Omega))$, $u_0\in H^1(\Omega)$, $u_1\in L^2(\Omega)$ and $f\in H^{1,1}(\Sigma)$. We assume that the compatibility condition $g(0,\cdot)=u_0|_\Gamma$ is satisfied. Then the IBVP* (A.1) *has a unique solution $u\in\mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u\in\mathcal{C}([0,T];L^2(\Omega))$ and $\partial_\nu u\in L^2(\Sigma)$. Moreover, we find a constant $C=C(T)$, not depending on F, u_0, u_1 and f, such that*

$$E_u(T)=\|u\|_{\mathcal{C}([0,T];H^1(\Omega))}+\|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))}+\|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\big\|(F,u_0,u_1,f)\big\|, \tag{A.2}$$

where

$$\big\|(F,u_0,u_1,f)\big\| = \|F\|_{L^1(0,T;L^2(\Omega))}+\|u_0\|_{H^1(\Omega)}+\|u_1\|_{L^2(\Omega)}+\|f\|_{H^{1,1}(\Sigma)}.$$

Next we introduce the following IBVP

$$\begin{cases}\partial_t^2 u-\Delta u+q(x)u=F & \text{in } Q,\\ u(0,\cdot)=0, \quad \partial_t u(0,\cdot)=0 & \text{in } \Omega,\\ u=0 & \text{on } \Sigma.\end{cases} \tag{A.3}$$

Theorem A.2. *Let $F \in L^2(Q)$ and $q \in L^\infty(\Omega)$. Then the IBVP* (A.2) *has a unique solution $u \in \mathcal{C}([0,T]; H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T]; L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$. Moreover, if $\|q\|_{L^\infty(\Omega)} \leqslant M$ with some positive constant M, then we find a constant $C = C(T, M)$, not depending on F and q, such that*

$$E_u(T) = \|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\|F\|_{L^1(0,T;L^2(\Omega))}.$$

Proof. First, from Theorems 8.1 and 8.3 in Lions and Magenes [18], we deduce that (A.3) has a unique solution $u \in \mathcal{C}([0,T]; H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T]; L^2(\Omega))$. Since u is also the solution of (A.1) with $G = F - qu$ in place en F, $u_0 = 0$, $u_1 = 0$ and $f = 0$, we derive from Theorem A.1 that we have also $\partial_\nu u \in L^2(\Sigma)$.

In the sequel we shall use the following notation $Q_s = (0,s) \times \Omega$.

Let $0 < S < T$ and let $\chi_{(0,S)}(t) = 1$ if $0 < t < S$ and 0 otherwise. Let $\widetilde{G} = \chi_{(0,S)} G$, we denote by $\tilde{u}$ the solution of (A.1) with F changed by $\widetilde{G}$. It follows from estimate (A.2) that

$$E_{\tilde{u}}(S) \leqslant CE_{\tilde{u}}(T) \leqslant \|\widetilde{G}\|_{L^1(0,T;L^2(\Omega))} = \|F - qu\|_{L^1(0,S;L^2(\Omega))}.$$

By the uniqueness of the solutions of (A.1) in Q_S, we have $\tilde{u} = u$ in Q_S. Therefore

$$E_u(S) \leqslant C\|F - qu\|_{L^1(0,S;L^2(\Omega))} \leqslant C\big(\|F\|_{L^1(0,T;L^2(\Omega))} + \|q\|_{L^\infty(\Omega)}\|u\|_{L^1(0,S;L^2(\Omega))}\big).$$

That is,

$$\|u\|_{L^1(0,S;L^2(\Omega))} \leqslant S\|u\|_{C([0,T];H^1(\Omega))} \leqslant SE_u(S).$$

Hence

$$E_u(S) \leqslant C\|F\|_{L^1(0,T;L^2(\Omega))} + KSE_u(S),$$

where $K = CM$. We assume that $S < 1/(2K)$. Then

$$E_u(S) \leqslant 2C\|F\|_{L^1(0,T;L^2(\Omega))}. \tag{A.4}$$

In particular, we have

$$\big\|u(\cdot,S)\big\|_{H^1(\Omega)} + \big\|\partial_t u(\cdot,S)\big\|_{L^2(\Omega)} \leqslant 2C\|F\|_{L^1(0,T;L^2(\Omega))}. \tag{A.5}$$

Let $v(x,t) = u(x, t+S)$, $(x,t) \in Q_{T-S}$. Clearly, v is the solution of IBVP (A.1) with F replaced by $F(x,t+S) - q(x)v(x,t)$, $u_0 = u(\cdot,S)$ and $v_0 = \partial_t u(\cdot,S)$. Repeating the same argument as before, we find

$$E_v(S) \leqslant C\big(\|F\|_{L^1(0,T;L^2(\Omega))} + \big\|u(\cdot,S)\big\|_{H^1(\Omega)} + \big\|\partial_t u(\cdot,S)\big\|_{L^2(\Omega)}\big) + KSE_v(S).$$

This and (A.5) yield

$$E_v(S) \leqslant 3C\|F\|_{L^1(0,T;L^2(\Omega))} + KSE_v(S).$$

Since $S < 1/(2K)$, we obtain

$$E_v(S) \leqslant 6C\|F\|_{L^1(0,T;L^2(\Omega))}.$$

This estimate in combination with (A.4) gives

$$E_u(2S) \leqslant E_u(S) + E_v(S) \leqslant 8C\|F\|_{L^1(0,T;L^2(\Omega))}.$$

We pursue this argument until $T - pS < 1/(2K)$ for some positive integer p. Then we make a final step between pS and T. We derive the following estimate

$$E_u(T) \leqslant mC\|F\|_{L^1(0,T;L^2(\Omega))},$$

where m is a positive integer depending on T and M. $\square$

We will now combine the previous two theorems to obtain a version of Theorem A.1 for the following IBVP

$$\begin{cases} \partial_t^2 u - \Delta u + q(x)u = F & \text{in } Q, \\ u(0,\cdot) = u_0, \quad \partial_t u(0,\cdot) = u_1 & \text{in } \Omega, \\ u = f & \text{on } \Sigma. \end{cases} \tag{A.6}$$

Theorem A.3. *Let $q \in L^\infty(\Omega)$ and $M \geqslant \|q\|_{L^\infty(\Omega)}$ with positive constant M. If $F \in L^1(0,T;L^2(\Omega))$, $u_0 \in H^1(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in H^{1,1}(\Sigma)$, and the compatibility condition $g(0,\cdot) = u_0|_\Gamma$ is satisfied, then IBVP* (A.3) *has a unique solution $u \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_\nu u \in L^2(\Sigma)$. Moreover, we find a constant $C = C(T,M)$, not depending on F, u_0, u_1 and f, such that*

$$E_u(T) = \|u\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t u\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu u\|_{L^2(\Sigma)} \leqslant C\left\|(F,u_0,u_1,f)\right\|, \tag{A.7}$$

where

$$\left\|(F,u_0,u_1,u_1,f)\right\| = \|F\|_{L^1(0,T;L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} + \|f\|_{H^{1,1}(\Sigma)}.$$

Proof. Let v be the solution of (A.1). Then $v \in \mathcal{C}([0,T];H^1(\Omega))$, $\partial_t v \in \mathcal{C}([0,T];L^2(\Omega))$, $\partial_\nu v \in L^2(\Sigma)$ and the following estimate holds

$$E_v(T) = \|v\|_{\mathcal{C}([0,T];H^1(\Omega))} + \|\partial_t v\|_{\mathcal{C}([0,T];L^2(\Omega))} + \|\partial_\nu v\|_{L^2(\Sigma)} \leqslant C_0\left\|(F,u_0,u_1,f)\right\|, \tag{A.8}$$

where $C = C(T)$ is a constant not depending on F, u_0, u_1 and f. We consider then the following IBVP

$$\begin{cases} \partial_t^2 w(t,x) - \Delta w(t,x) + q(x)w(t,x) = -q(x)v(t,x) & \text{in } Q, \\ w(0,x) = 0, \quad \partial_t w(0,x) = 0 & \text{in } \Omega, \\ w(t,x) = 0 & \text{on } \Sigma. \end{cases} \tag{A.9}$$

It follows from Theorem A.2 that (A.9) has a unique solution $w \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t w \in \mathcal{C}([0,T];L^2(\Omega))$, $\partial_\nu w \in L^2(\Sigma)$ and we have the following estimate

$$E_w(T) \leqslant C_0\|qv\|_{L^1(0,T;L^2(\Omega))} \leqslant C_0 MT E_v(T). \tag{A.10}$$

One can then easily see that $u = v + w$ is the unique solution of (A.6) with $u \in \mathcal{C}([0,T];H^1(\Omega))$ such that $\partial_t u \in \mathcal{C}([0,T];L^2(\Omega))$, $\partial_\nu u \in L^2(\Sigma)$. Moreover a combination of (A.8) and (A.10) yields estimate (A.7). $\square$

References

[1] G. Alessandrini, J. Sylvester, Stability for multidimensional inverse spectral problem, Comm. Partial Differential Equations 15 (5) (1990) 711–736.

[2] G. Bao, K. Yun, On the stability of an inverse problem for the wave equation, preprint.

[3] M. Belishev, Boundary control in reconstruction of manifolds and metrics (BC method), Inverse Problems 13 (1997) R1–R45.

[4] M. Bellassoued, Global logarithmic stability in inverse hyperbolic problem by arbitrary boundary observation, Inverse Problems 20 (2004) 1033–1052.

[5] M. Bellassoued, D. Jellali, M. Yamamoto, Lipschitz stability for a hyperbolic inverse problem by finite local boundary data, Appl. Anal. 85 (2006) 1219–1243.

[6] M. Bellassoued, D. Jellali, M. Yamamoto, Stability estimate for the hyperbolic inverse boundary value by local Dirichlet-to-Neumann map, J. Math. Anal. Appl. 343 (2008) 1036–1046.

[7] F. Cardoso, R. Mendoza, On the hyperbolic Dirichlet to Neumann functional, Comm. Partial Differential Equations 21 (1996) 1235–1252.

[8] M. Choulli, Une Introduction aux Problèmes Inverses Elliptiques et Paraboliques, Math. Appl. (Berlin), Springer, Berlin, 2009, in press.

[9] R. Courant, D. Hilbert, Methods of Mathematical Physics, Interscience, New York, 1962.

[10] R. Cipolatti, Ivo F. Lopez, Determination of coefficients for a dissipative wave equation via boundary measurements, J. Math. Anal. Appl. 306 (2005) 317–329.

[11] R. Dautray, J.L. Lions, Analyse Mathématique et Calcul Numérique, vol. V, Masson, Paris, 1985.

[12] V. Isakov, An inverse hyperbolic problem with many boundary measurements, Comm. Partial Differential Equations 16 (1991) 1183–1195.

[13] V. Isakov, Z. Sun, Stability estimates for hyperbolic inverse problems with local boundary data, Inverse Problems 8 (1992) 193–206.

[14] A. Katchalov, Y. Kurylev, M. Lassas, Inverse Boundary Spectral Problems, Chapman & Hall/CRC, Boca Raton, 2001.

[15] Y.V. Kurylev, M. Lassas, Hyperbolic inverse problem with data on a part of the boundary, in: Differential Equations and Mathematical Physics, in: AMS/IP Stud. Adv. Math., vol. 16, Amer. Math. Soc., Providence, 2000, pp. 259–272.

[16] I. Lasiecka, J.-L. Lions, R. Triggiani, Non homogeneous boundary value problems for second order hyperbolic operators, J. Math. Pures Appl. 65 (1986) 149–192.

[17] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1995) 335–356.

[18] J.L. Lions, E. Magenes, Problèmes aux Limites Non Homogènes et Applications, vol. I, Dunod, Paris, 1968.

[19] J.L. Lions, E. Magenes, Problèmes aux Limites Non Homogènes et Applications, vol. II, Dunod, Paris, 1968.

[20] F. Natterer, The Mathematics of Computerized Tomography, John Wiley & Sons, Chichester, 1986.

[21] Rakesh, Reconstruction for an inverse problem for the wave equation with constant velocity, Inverse Problems 6 (1990) 91–98.

[22] Rakesh, W. Symes, Uniqueness for an inverse problems for the wave equation, Comm. Partial Differential Equations 13 (1988) 87–96.

[23] L. Robbiano, Théorème d'unicité adapté au contrôle des solutions des problèmes hyperboliques, Comm. Partial Differential Equations 16 (1991) 789–800.

[24] L. Robbiano, C. Zuily, Uniqueness in the Cauchy problem for operators with partially holomorphic coefficients, Invent. Math. 131 (1998) 493–539.

[25] Z. Sun, On continuous dependence for an inverse initial boundary value problem for the wave equation, J. Math. Anal. Appl. 150 (1990) 188–204.